EXTENDING BEYOND CONDITIONAL RANDOM FIELDS USING BOOLEAN RINGS OF BLACKENING OPERATORS

Samuel Guilluy
IRMAR
Université Rennes 1
Rennes
samuel.guilluy@univ-rennes1.fr

Florian Méhats
IRMAR
Ravel Technologies
on leave from Univ Rennes
Rennes
florian.mehats@univ-rennes1.fr

Billal Chouli
Headmind Partners AI & Blockchain
Paris
bchouli812@headmind.com

ABSTRACT

This article introduces a novel algebraic framework designed to harness data structures and relational dependencies. Specifically, we aim to identify the optimal factorization of the joint probability function while accommodating dependency constraints that are less restrictive than traditional Markovian dependencies. In this context, our approach can be considered as an extension of Hammersley and Clifford's random field theorem, using the principles of Boolean rings and principal ideals. In addition to presenting our new method, we present a new proof for the fundamental theorem of random fields. Finally, we prove the practical effectiveness of our methodology through an application in argument mining. Our findings highlight the potential of this framework in various data-driven applications.

***K*eywords** Markov Random Fields · Boolean Algebra · Artificial Intelligence

1 Introduction

The field of graph structured data is increasingly important in a variety of fields, including computational biology [1], natural language processing [2], and social science [3]. One of the main challenges in this area is understanding the underlying structure of the data and the analysis of the functions defined on these data, as highlighted by [4] in their 2017 paper on geometric deep learning.

Markov Random Fields, first introduced by Hammersley and Clifford in [5], have been a fundamental tool for understanding data structures, providing factorization formulae for Markovian lattices. However, these models have limitations in terms of expressiveness, due to the strong assumption on dependencies between neighbours, and recent studies [6] [7] have shown the limitations of Graph Neural Networks (GNNs) as well.

The objective of this paper is to propose an algebraic framework to the problem of finding the best neural network architecture given the underlying data structure and dependency relationships involved. We provide a new methodology for exploiting data structure and relational dependency, which relaxes the Markovian assumption of Conditional Random Fields (CRFs). Specifically, we exploit the property of principal ideals in a Boolean ring using the theory of Boolean rings presented by Stone in [8] to find the best factorization of the join probability function that respects dependency constraints.

We begin by introducing the theory of Boolean rings and the concept of blackening operators in order to prove a factorization result. As a consequence, the joint probability function in a graph can be factorized under some dependency constraints. This approach is intended to serve as a novel and general methodology for exploiting data structures and relational dependencies in a graph. To further showcase the effectiveness of our proposed approach, we provide an example of its application in the context of argument mining.

2 Related works

Boolean algebra of projectors, as introduced by [8] in his representation theorem for Boolean algebras, provides a powerful tool for understanding data structures. Stone's theorem states that every Boolean algebra is isomorphic to a certain field of sets, and this result has been widely used in various fields such as the spectral theory of operators on a Hilbert space ([9]), Boolean algebra of projectors ([10]) ([11]) and lattices structures ([12] and [13]). These results also lead to many results in category theory related to topological space ([14]).

Markov Random Fields (MRFs) have been a fundamental tool for understanding data structures since their introduction in [5]. They provide a way to factorize graphs under the Markovian assumption, and have been successfully applied in various fields such as Natural Language Processing in [15] and [16] and Computer Vision in [17]. However, MRFs have limitations in terms of expressiveness due to the strong assumption of dependencies between neighbours, which has been highlighted in previous research studies : [18], [19] and [20].

In addition, working with oriented graphs poses several additional challenges, particularly in terms of the representation and analysis of their underlying structure. Many efforts have been made to address these challenges, including the use of Graph Neural Networks (GNNs) with architectures such as BiGraphSAGE [21], LSTM [22], and Neural Trees [23]. These architectures aim to exploit the tree structure present in directed graphs, allowing for more accurate and efficient representations of the graph.

3 The Boolean ring of blackening operators

3.1 Preliminaries: Boolean rings and ideal factorization

The main goal of this subsection is to provide an algebraic framework for the rest of the paper. More precisely, we define conditions under which unions and intersections of certain ideals can be reduced into more compact forms. An abstract formulation of the $I(\beta)$ used in [5] for the CRF proof is stated and proved in Theorem 1 using elementary results from the theory of Boolean rings and principal ideals. This theorem will then be applied in Section 3.2.

Definition 1 *Let A be a commutative ring for the operations $\oplus$ and $\otimes$ (with neutral elements denoted by 0 and $\mathbb{1}$). We say that A is a* Boolean ring *if any element $a \in A$ is idempotent, i.e. $a \otimes a = a$. On the Boolean ring A, we shall consider the partial order $\leq$ defined by*

$$\forall (b,c) \in A^2, \qquad b \leq c \quad \textit{if, and only if,} \quad b \otimes c = b. \tag{1}$$

A standard result on Boolean rings is that any subset of the ring admits a least upper bound $\vee$ and a greatest lower bound $\wedge$. For all $a, b \in A$, one has

$$a \vee b = a \oplus b \oplus a \otimes b, \qquad a \wedge b = a \otimes b.$$

In addition, the complementary of $a \in A$ is denoted by $a' = \mathbb{1} \oplus a$. Note that the set A, equipped with $\vee$, $\wedge$ and the complementary, is endowed with a Boolean algebra structure, see e.g. [24]. Conversely on any Boolean algebra A, it can be defined two operations $\oplus$ and $\otimes$ such that A is a Boolean ring. For completeness, more details on Boolean algebras and Boolean rings and given in supplementary material A.

Definition 2 *A non empty subset I of a Boolean ring is an* ideal *if, and only if,*

- *I is closed under the addition:* $\quad \forall (a,b) \in I^2, \quad a \oplus b \in I;$
- *I is stable with respect to the partial order $\leq$: $\forall (a,c) \in I \times A$, $c \leq a$ implies that $c \in I$.*

The ideal generated by $a \in A$ is defined as $I(a) = \{b \in A,\ b \leq a\}$. An ideal is said to be principal if it is generated by one of its elements.

We also need to introduce the definition of the orthogonal of a subclass of a Boolean ring.

Definition 3 *Two elements a and b in a Boolean ring are said to be orthogonal if $a \otimes b = 0$. Two non-void subclasses of a Boolean ring are said to be orthogonal if every element of one is orthogonal to every element of the other. We denote by $I(a)^{\perp}$ the class of all elements orthogonal to every element of $I(a)$.*

Defining finally the addition of subsets of a Boolean ring as $I + J = \{i \oplus j \mid i \in I \; and \; j \in J\}$, we are now able to state and prove the main theorem of this section.

Theorem 1 *Let $(a_j)_{j\in J}$ and $(b_j)_{j\in J}$ be two sets of elements of a Boolean ring A. Then we have*

$$I\left(\prod_{j\in J}(a_j \vee b_j)\right) = \sum_{K\subset \mathcal{P}(J)} \bigcap_{k_1\in K} I(a_{k_1}) \bigcap_{k_2\in J\setminus K} I(b_{k_2}) \bigcap_{k_3\in J\setminus K} I(a'_{k_3}),$$

where $\mathcal{P}(J)$ is the set of all the subsets of J and $\sum$ is the addition and $\bigcap$ the intersection on sets.

Before proving this theorem, we recall Theorem 31 of [8].

Lemma 1 *The class $\mathbb{P}$ of all principal ideals in a Boolean ring A is isomorphic to the Boolean ring A itself in accordance with the following relations :*

1. $I(a) = I(b)$ *if and only if* $a = b$.
2. $I(a \oplus b) = I(a) + I(b) = (I(a)I(b)^{\perp}) \cup I(a)^{\perp}I(b)$.
3. $I(a \vee b) = I(a) \cup I(b), \quad I(a \wedge b) = I(a) \cap I(b) \quad and \quad I(a') = I(a)^{\perp}$.

Proof 1 (Proof of Theorem 1) *We first deduce from Lemma 1 that*

$$\begin{aligned} I(a \vee b) &= I(a \oplus b \oplus a \otimes b) = I(a \oplus b \otimes (\mathbb{1} \oplus a)), \\ &= I(a \oplus b \otimes a') = I(a) + I(b \otimes a'), \\ &= I(a) + I(b) \cap I(a') = I(a) + I(b) \cap I(a)^{\perp}. \end{aligned} \tag{2}$$

We can now develop, using again Lemma 1,

$$\begin{aligned} I\left(\prod_{j\in J}(a_j \vee b_j)\right) &= \bigcap_{j\in J}(I(a_j) + I(b_j) \cap I(a_j)^{\perp}), \\ &= \sum_{K\subset P(J)} \bigcap_{k_1\in K} I(a_{k_1}) \bigcap_{k_2\in J\setminus K} I(b_{k_2}) \cap I(a_{k_2})^{\perp}, \\ &= \sum_{K\subset P(J)} \bigcap_{k_1\in K} I(a_{k_1}) \bigcap_{k_2\in J\setminus K} I(b_{k_2}) \bigcap_{k_3\in J\setminus K} I(a'_{k_3}), \end{aligned} \tag{3}$$

which yields the result.

To clarify this proof, we take the sum over all the possible subsets of J. We denote K as one of the given subsets. As proved in Equation 3, when we take the product of a sum, we take all the possible combination which means that there is a subset of nodes K where the chosen term is $I(a_{k_1})$ and the other $J\setminus K$ is for $I(b_{k_2})$ inter $I(a'_{k_3})$.

To apply this result to graphs, we now need to introduce a framework in relation with graphs and compatible with Boolean rings.

3.2 Definition of the polynomial blackening operators

The main goal of this subsection is the construction of the pure blackening operators and the polynomials operators.

Definition 4 *Let G be a (oriented or not) graph with $Z = \{z_i\}$ the set of nodes of G and $E = (z_i, z_j)$ the set of the edges of G. We define the* colors *C as a finite set of elements $\{c_j\}$ containing the color "black". A* coloration *of a graph G is an application χ from Z to C. The set of colorations will be denoted by $\mathcal{C}$.*

For all coloration χ, we denote by χ_Y the application that attributes the same color as χ to any node of the graph except for the set of nodes Y which are blacken. In particular, χ_Z corresponds to a totally black coloration of the graph.

We now consider the set $\mathcal{F}$ of real-valued functions defined on the colorations $\mathcal{C}$ of the graph G and which attribute the value zero to χ_Z. Our main quantity of interest will be the set of operators on $\mathcal{F}$, on which we first define three operations $\vee$, $\wedge$ and $'$. We denote by $\mathbb{1}$ and 0 respectively the identity operator and the null operator.

Definition 5 *Let P and Q be two operators on $\mathcal{F}$, we define:*

$$P \vee Q = P + Q - P \circ Q, \quad P \wedge Q = P \circ Q, \quad \neg P = \mathbb{1} - P, \tag{4}$$

where $+$ and $-$ are induced by the corresponding operations on $\mathbb{R}$ and $\circ$ is the composition operator.

Now we consider a specific class of operators on $\mathcal{F}$ (illustrated below on Figure 1) and the ring generated by this class.

Definition 6 *Considering a subset Y of Z, we define the operator B_Y, called the* pure blackening operator*, as the following operator on $\mathcal{F}$:*

$$\forall F \in \mathcal{F}, \forall \chi \in \mathcal{C}, \qquad (B_Y F)(\chi) = F(\chi_Y). \tag{5}$$

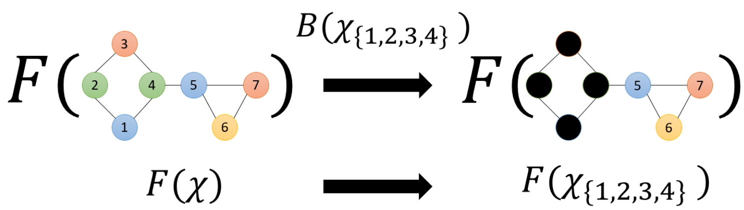

Figure 1: Illustration of the action of a pure blackening operator on a function F where the set of nodes $\{1, 2, 3, 4\}$ are blackened. The input graph of the function is partially blackened.

Definition 7 *Let $\{X_i\}_{i\in[1:n]}$ and $\{Y_j\}_{j\in[1:m]}$ be two finite sets of subsets of Z. We define the* monomial blackening operator *$M_{X,Y}$ associated with the sets $\{X_i\}_{i\in[1:n]}$ and $\{Y_j\}_{j\in[1:m]}$ as*

$$M_{X,Y} = \underset{1\leq i\leq n}{\wedge} B_{X_i} \underset{1\leq j\leq m}{\wedge} (\neg B_{Y_j}).$$

Let $\{M_{X^a,Y^a}\}_{a\in[1:\ell]}$ be a set of monomial blackening operators, we define the polynomial blackening operator *P as*

$$P = \underset{1\leq a\leq \ell}{\vee} M_{X^a,Y^a}.$$

We denote the set of monomial blackening operators *as $\mathcal{M}$ and the set of* polynomials operators *as $\mathcal{P}$.*

3.3 Characteristics of pure blackening operators and polynomials operators

In this subsection, we presents some results about operations over pure blackening operators and polynomials operators which will be required in the next subsection to construct the Boolean algebra of polynomial blackening operators.

Lemma 2 *Let W, X and Y be three subsets of Z. The pure blackening operators B_W, B_X, B_Y have the following properties*

- *Complement: $(\mathbb{1} - B_X)$ is the unique operator B satisfying*

$$B_X \vee B = \mathbb{1} \quad and \quad B_X \wedge B = 0. \tag{6}$$

  We will denote the complement of an operator B as $\neg B = (\mathbb{1} - B)$. We have the two De Morgan's laws:

$$\neg(B_X \vee B_Y) = \neg B_X \wedge \neg B_Y \quad and \quad \neg(B_X \wedge B_Y) = \neg B_X \vee \neg B_Y.$$

- *Commutativity:*
  - *For $\wedge$: $B_W \wedge B_X = B_X \wedge B_W \quad and \quad B_W \wedge \neg B_X = \neg B_X \wedge B_W$,*
  - *For $\vee$:*

$$B_W \vee B_X = B_X \vee B_W \quad and \quad B_W \vee \neg B_X = \neg B_X \vee B_W.$$

- *Associativity:*

$$(B_W \wedge B_X) \wedge B_Y = B_W \wedge (B_X \wedge B_Y) \quad and \quad (B_W \vee B_X) \vee B_Y = B_W \vee (B_X \vee B_Y).$$

- *Distributivity:*
$$B_W \wedge (B_X \vee B_Y) = (B_W \wedge B_X) \vee (B_W \vee B_Y).$$

The proof of this lemma can be found in Proof A.2 in the Appendix.

Lemma 3 *Let $\mathcal{P}$ be the set of* polynomials operators*. We have the following properties :*

1. *$\mathcal{P}$ is stable by $\wedge, \vee$ and $\neg$.*
2. *The elements of $\mathcal{P}$ commutes two by two:*
$$\forall (P, Q) \in \mathcal{P}^2, \qquad P \wedge Q = Q \wedge P.$$
3. *Every element of $\mathcal{P}$ is a projector:*
$$\forall P \in \mathcal{P}, \qquad P \wedge P = P.$$

The proof of this lemma can be found in Proof A.3 in the Appendix.

3.4 The Boolean algebra of polynomial blackening operators

The main goal of this subsection is the construction of the Boolean algebra of polynomial blackening operators. We will first present the structure, then apply Theorem 1.

Definition 8 *We define the relation $\leq_{\mathcal{P}}$ on the set $\mathcal{P}$ as follows.*

$$For\ all\ P, Q \in \mathcal{P}^2,\ P \leq_{\mathcal{P}} Q \quad \text{if, and only if,} \quad P \wedge Q = Q \quad \text{which is equivalent to} \quad P \vee Q = P$$

Lemma 4 *The set $(\mathcal{P}, \leq_{\mathcal{P}})$ is a partially ordered set and the operations $\vee$ and $\wedge$ in Definition 8 are respectively the least upper bound and the greatest lower bound as defined at the beginning of the Section. Moreover, $\neg$ is the complementary in the sense of Boolean Algebra.*

The proof of Lemma 4 can be found in Proof A.4 in the Appendix. In the sequel, we will denote $\leq_{\mathcal{P}}$ as $\leq$ without ambiguity.

Proposition 1 *The set of polynomial blackening operators $\mathcal{P}$ with the relation $\wedge$, $\vee$ and $\neg$ is a Boolean algebra. It is the smallest Boolean algebra containing the pure blackening operators. Moreover B_Z is the neutral element of $\mathcal{P}$ for the addition and $B_\emptyset$ is the identity element of $\mathcal{P}$ (neutral element for $\wedge$).*

The proof of Proposition 1 can be found in Proof A.5 in the Appendix. As a consequence and according to Subsection 3.1, one can define two operations $\oplus$ and $\otimes$ on $\mathcal{P}$ such that it is a Boolean ring.

Let us now apply Theorem 1, considering the union and intersection of pure blackening operators.

Corollary 1 *With the same notations as in Theorem 1, consider a subset $K \subset J$. Let $\{X_k\}$ and $\{Y_k\}$ be two subsets of $P(J)$ indexed by k and denote $a_k = B_{X_k}$ and $b_k = B_{Y_k}$. We identify necessary conditions in order to the term $\left(\bigcap_{k_1 \in K} I(a_{k_1}) \bigcap_{k_2 \in J \setminus K} I(b_{k_2}) \bigcap_{k_3 \in J \setminus K} I(a'_{k_3})\right)$ is not equal to $\{0\}$:*

1. *$\bigcup_{k \in K} X_k \neq Z$ and $\bigcup_{j \in (J \setminus K)} Y_j \neq Z$.*
2. *$\forall j \in (J \setminus K)$, $\quad X_j \not\subset \bigcup_{j' \in (J \setminus K)} Y_{j'}$.*
3. *$\forall j \in J \setminus K, \forall k \subset K, \quad X_j \not\subset X_k$.*
4. *$\forall j \in J \setminus K, \forall k \subset K, \quad X_k \cup Y_j \neq Z$.*
5. *$\forall (j, j') \in (J \setminus K)^2, \quad (\mathbb{1} - B_{X_j})(\mathbb{1} - B_{X_{j'}}) \neq 0$.*

This is a direct application of Theorem 1, the complete proof can be found in supplementary material A.6. This corollary will be our main tool for the reduction of principal ideals associated to the Boolean algebra of the polynomial blackening operators.

4 A new proof of the fundamental theorem of random fields

Before we explore a practical example, it is essential to begin with a new proof of the so-called fundamental theorem of random fields of Hammersley and Clifford [5]. This showcases the broad applicability and generality of our algebraic framework compared to the fundamental theorem of random fields.

Definition 9 *Let χ be a coloration of the graph G. We denote by the $\mathbb{P}(\omega = \chi)$ (in short $\mathbb{P}(\chi)$) the probability that a random coloring ω matches χ. Moreover, for all $Y \subset Z$, we denote by $\mathbb{P}(\chi^Y)$ the probability that the restriction of ω on Y matches the restriction of χ on Y. With the notation $C_{X,\chi} = \{\xi \in C \mid \forall z_i \in X, \xi(z_i) = \chi(z_i)\}$,*

$$\mathbb{P}(\chi^X) = \mathbb{P}(\omega|_X = \chi|_X) = \sum_{\xi \in C_{X,\chi}} \mathbb{P}(\xi).$$

Furthermore, for all $X \subset Z$, $Y \subset Z$, $\mathbb{P}(\chi^X, \chi^Y)$ is the probability that ω simultaneously has the partial colouring $\chi|_X$ on X as well as the partial colouring $\chi|_Y$ on Y. In addition, we note $\mathbb{P}(\chi^X|\chi^Y)$ the probability that the random colouring ω matches the specified colouring χ on the set X knowing that ω as the colouring $\chi|_Y$ on the set Y.

$$\mathbb{P}(\chi^X|\chi^Y) = \frac{\sum_{\xi \in C^{X\cup Y,\chi}} \mathbb{P}(\xi)}{\sum_{\xi \in C^{Y,\chi}} \mathbb{P}(\xi)} = \frac{\mathbb{P}(\chi^{X\cup Y})}{\mathbb{P}(\chi^Y)}. \tag{7}$$

Definition 10 *A random variable is said to be* globally Markovian *if it is Markovian for every subsets of Z. With the notation previously introduced, the Markovian assumption can be formulated as*

$$\forall X \subset Z, \quad \mathbb{P}(\chi^X|\chi^{Z\setminus X}) = \mathbb{P}(\chi^X|\chi^{\partial X}), \tag{8}$$

where ∂X is the set of all the neighbours of the nodes of X.

$$\partial X = \{z_j \in Z\setminus X \mid \exists z_i \in X, \quad (z_i, z_j) \in E \quad or \quad (z_j, z_i) \in E\},$$

where E is the set of edges of G.

Definition 11 *We define a* clique *of a graph as a set of nodes where each node is neighbour of each other. We denote by L the set of cliques of G. Moreover, given a coloration χ of the graph, we define a* light clique *as a clique where every node is not black. We denote this set $L(\chi)$.*

Theorem 2 (Hammersley-Clifford's theorem [5]) *Let ω be a random coloring of Z which follows the globally Markovian properties. Suppose that $\mathbb{P}(\omega = \chi_Z) \neq 0$. Then there exist $S \subset \mathcal{F}$ such that we can factorize the probability as follows:*

$$\mathbb{P}(\chi) = \mathbb{P}(\chi_Z) \exp\left(\sum_{Y \subset L(\chi)} S(\chi_{Z\setminus Y})\right). \tag{9}$$

Our reformulation of Theorem 2 consists in identifying a specific principal ideal in the Boolean ring of Blackening operators and reducing it. Let us introduce the specific principal ideal $I(\beta)$.

Definition 12 *Let Z be the set of nodes of a graph G and for all node $z_i \in Z$, let ∂z_i the set of all its neighbours. We define the polynomial operator β_i as follows:*

$$\beta_i = B_{z_i} \vee B_{Z\setminus\{z_i \cup \partial z_i\}}.$$

For $X \subset Z$, we define

$$\beta_X = B_X \vee B_{Z\setminus\{X \cup \partial X\}}.$$

And we also define β as the product of all the β_i,

$$\beta = \prod_{z_i \in Z} \beta_i = \prod_{z_i \in Z} (B_{z_i} \vee B_{Z\setminus(z_i \cup \partial z_i)}).$$

Let us state two technical lemmas. We first reduce the formulation of $I(\beta)$ using Corollary 1.

Lemma 5 *$I(\beta)$ can be decomposed as :*

$$I(\beta) = \sum_{X \in L(\chi) \cup \{\emptyset\}} I(B_{Z \setminus X}), \tag{10}$$

where $L(\chi)$ is the set of light cliques of the graph G associated to the coloration χ.

The proof of this lemma can be found in Proof A.7 in the Appendix.

Lemma 6 *Let χ be a coloration of the set of nodes Z. Suppose that ω is globally Markovian. For all $X \subset Z$, we introduce the function*

$$Q_X = \log(\mathbb{P}(\chi^X)).$$

Then we have

$$\forall X \subset Z, \quad Q_Z = \beta_X Q_Z.$$

The proof of this lemma can be found in Proof A.8. Using Lemma 5 and Lemma 6, we are now able to provide a new proof of Hammersley-Clifford's Theorem.

Proof 2 (Proof of Theorem 2) *Let G be a graph and χ be a coloration on the graph G. Let ω be a random coloring of Z which follows the globally Markovian properties. Suppose that $\mathbb{P}(\omega = \chi_Z) \neq 0$. Using Lemma 6 and remarking that $Q_Z = \log(\mathbb{P}(\chi))$, we get*

$$\forall z_i \in Z, \quad \beta_i \log(\mathbb{P}(\chi)) = \log(\mathbb{P}(\chi)).$$

Hence,

$$\beta \log(\mathbb{P}(\chi)) = \prod_{z_i \in Z} \beta_i \log(\mathbb{P}(\chi)) = \log(\mathbb{P}(\chi)).$$

Thus, $\beta \log(\mathbb{P}(\chi)) = \log(\mathbb{P}(\chi))$.

Moreover, by Lemma 5, $\beta \in I(\beta) = \sum_{X \in L(\chi) \cup \{\emptyset\}} I(B_{Z \setminus X})$. Thus, there exists a set of projectors $E_{Z \setminus X}$ such that $E_{Z \setminus X} \in I(B_{Z \setminus X})$ and

$$\beta = \sum_{X \in L(\chi) \cup \{\emptyset\}} E_{Z \setminus X} = \sum_{X \in L(\chi) \cup \{\emptyset\}} B_{Z \setminus X} E_{Z \setminus X}, \tag{11}$$

$$\begin{aligned} \log(\mathbb{P}(\chi)) = \beta \log(\mathbb{P}(\chi)) &= \sum_{X \in L(\chi) \cup \{\emptyset\}} B_{Z \setminus X} E_{Z \setminus X} \log(\mathbb{P}(\chi)), \\ &= \sum_{X \in L(\chi) \cup \{\emptyset\}} E_{Z \setminus X} \log(\mathbb{P}(\chi_{Z \setminus X})). \end{aligned} \tag{12}$$

Finally, as $\mathbb{P}(\omega = \chi_Z) \neq 0$, with the notation $S(\chi_{Z \setminus X}) = E_{Z \setminus X} \log(\mathbb{P}(\chi_{Z \setminus X}))$, we have the final result:

$$\mathbb{P}(\chi) = \mathbb{P}(\chi_Z) \exp(\sum_{X \in L(\chi)} S(\chi_{Z \setminus X})). \tag{13}$$

5 Methodology

In this section, we present a strategy to reduce deep learning models on graphs. We first sketch the main steps of this strategy, then we explain why this method works by explaining the link between propositional logic and random variables respecting dependency constraints. An example of application will be provided in Section 6, where this methodology is exemplified on tree structure dependencies.

5.1 Description of the methodology

We now propose a methodology that generalized the above proof of the fundamental theorem of random fields. This methodology splits into four steps, using the results of Section 3.2 and is aimed to develop new machine learning architectures. The notion of dependency between nodes is essential in this methodology. We shall say that a node z_i is independent of another node z_j if, and only if, z_j can be blackened independently of z_i. The strategy can be summarized as follows.

x	y	$x \cap y$	$x \cup y$
False	False	False	False
True	False	False	True
False	True	False	True
True	True	True	True

Table 1: Truth table for binary operators.

- **Step 1: Identification of the invariance properties.** Identify the dependency relationships between the nodes in terms of local invariance properties. The goal of this first step is to identify locally which nodes are independent of which other nodes.
- **Step 2: Construction of the associated blackening operators.** For each node z_i of the graph G, formulate the invariance properties identified in Step 1 in terms of invariance under a specific blackening operator β.
- **Step 3: Link to probability function.** Prove that for a random coloration ω satisfying the invariance properties identified in Step 1, there holds
$$\beta \log(\mathbb{P}) = \log(\mathbb{P}).$$
- **Step 4: Reduction of the blackening operators.** Use a result similar to Corollary 1 to find the reduced form of the principal ideal generated by β.

5.2 Clarifying Step 2: leveraging the link between Boolean algebra and propositional logic

In this section we highlight a useful isomorphism between the Blackening algebra studied above and the two-element Boolean algebra associated with the fact that a function is invariant or not to some operators.

Definition 13 *We call the* two-element Boolean Algebra *the Boolean algebra*
$$(\{\text{True}, \text{False}\}, \cup, \cap, \text{False}, \text{True}),$$
where the operations are defined in the truth table 1.

Let us now define the propositional function ψ_F (in the sense of propositional logic) related to the invariance properties of elements of $\mathcal{F}$. The notations $\mathcal{F}$ and $\mathcal{P}$ were introduced in Section 3.

Definition 14 *For all function $F \in \mathcal{F}$ and $P \in \mathcal{P}$, we set*
$$\psi_F(P) = \text{True} \quad \textit{if} \;\; \mathrm{PF} = \mathrm{F} \quad \textit{and} \quad \psi_{\mathrm{F}}(\mathrm{P}) = \text{False} \quad \textit{otherwise}. \tag{14}$$

This function ψ_F takes its values in the two-elements Boolean algebra $\{\text{True}, \text{False}\}$ (equipped with the operations $\cup$ and $\cap$ associated to the classical truth table of binary logic). A key feature is that ψ_F is a morphism between the Boolean algebra $\mathcal{P}$ and this two-elements Boolean algebra.

Proposition 2 *For all $F \in \mathcal{F}$, ψ_F is a morphism in the following sense.*
For all $(P, Q) \in \mathcal{P}^2$, we have

1.
$$\psi_F(B_\emptyset) = \text{True}, \quad \text{and} \quad \psi_{\mathrm{F}}(\mathrm{B_Z}) = \text{False},$$

2.
$$\psi_F(P \vee Q) = \psi_F(P) \cup \psi_F(Q),$$

3.
$$\psi_F(P \wedge Q) = \psi_F(P) \cap \psi_F(Q),$$

4.
$$\psi_F(\neg P) = \neg \psi_F(P).$$

The proof of Proposition 2 can be found in Appendix A.9. This result explains why in Step 2 the construction of β_i based on two polynomial operators and the definition of β as the product of the β_i is fruitful.

In the next Section, we examplarize our strategy on a tree structure with non-symmetric dependency relations between nodes.

6 Application to argument mining

In this section, we provide a toy model in order to illustrate how the above theoretical framework can be applied. In the context of argument mining, we show how to leverage the additional information provided by annotation schemes in supervised learning tasks. Indeed in the field of Natural Language Processing (NLP), many tasks depend on the coherence with the grammatical structure of the sentence, which can be encoded as a directed graph.

In our experimentation (see Subsection 6.3), we investigate an argument mining task presented in [25] that involves identifying whether argumentative sentences are in favor of or against a given topic. The main challenge associated with this task is the difficulty in constructing a large dataset, which necessitates the use of pre-trained Large Language Models and fine-tuning them for our specific task.

Before dealing with our concrete application, we need to introduce a theoretical setting that fits into our methodology. It will enable us to design an efficient architecture for our argument mining model.

6.1 Definition of an arborescence and a filter in trees

Definition 15 *An* arborescence *is a directed graph G in which, for a specific node u (called the root) and any other node v, there is exactly one directed path (i.e. a sequence of edges) from u to v . We can view an arborescence as a directed rooted tree.*

Within an arborescence, we can introduce the concepts of children, parents and siblings of a node.

Definition 16 *The* children *(*parents*) of a node z_i is the set of nodes $C(z_i)$ (resp. $P(z_i)$) composed of all the nodes z_j where (z_i, z_j) (resp. (z_j, z_i)) is an edge of the arborescence.*

$$C(z_i) = \{z_j | (z_i, z_j) \in E\}, \quad P(z_i) = \{z_j | (z_j, z_i) \in E\}.$$

The descendants *$D(z_i)$ (resp.* ancestors *$A(z_i)$) of z_i are defined as the sets of the children (resp. parents) of z_i and the children (resp. parents) of its children (resp. parents) recursively. The* siblings *$Sib(z_i)$ of z_i are defined as the set of nodes which have the same parents as z_i.*

We extend this notions to any subset Y of the arborescence:

$$D(Y) = \left(\bigcup_{z_i \in Y} D(z_i) \right) \setminus Y, \quad A(Y) = \left(\bigcup_{z_i \in Y} A(z_i) \right) \setminus Y, \quad Sib(Y) = \left(\bigcup_{z_i \in Y} Sib(z_i) \right) \setminus Y.$$

One important notion that we will use in the sequel is the filter on an arborescence.

Definition 17 *A non-empty subset F of a partially ordered set Q is an ordered filter if the following conditions hold:*

- *F is downward directed: $\forall x, y \in F$, there exists $z \in F$ such that $z \leq x$ and $z \leq y$.*
- *F is an upper set: for every $x \in F$ and $p \in Q$, $\quad x \leq p$ implies that $p \in F$.*

In the case of a tree structure, every set of siblings leaves with their common ancestors (i.e. all ancestors up to the root of the tree) is a filter for the order induced by the direction of the edges. We denote by $F(G)$ the set of filters of the graph G.

To enhance the visual understanding of the concept of a clique, Figure 2 offers a comparative representation of the factorized structure of cliques in graphs and filters in trees.

6.2 Application of the methodology

Let us state the main result of this section.

Theorem 3 *Let ω be a random coloring of the set of nodes Z of an arborescence. Assume that the coloring follows the blackening consistent property (defined below in Step 1) and that $\mathbb{P}(\omega = \chi_Z) \neq 0$. We can factorize the probability as follows:*

$$\exists S \in \mathcal{F}, \quad \mathbb{P}(\chi) = \mathbb{P}(\chi_Z) \exp \left(\sum_{Y \subset F(G)} S(\chi_Y) \right). \tag{15}$$

In other words, the probability law of colorations can be factorized on the filters of the arborescence.

To prove this theorem, we follow Steps 1, 2, 3 and 4 of our methodology.

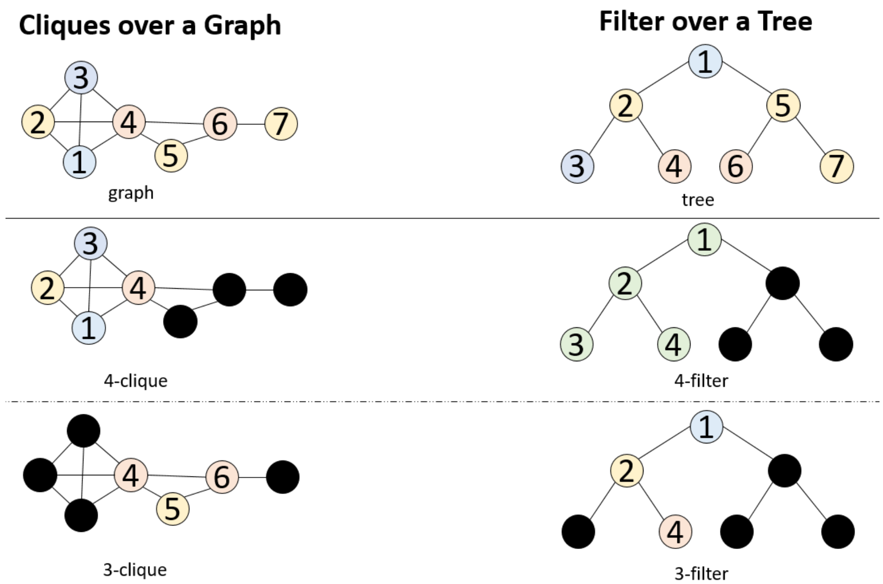

Figure 2: Illustration on the difference between the factorization over filter and the factorization over clique. The left side refers to identifying some cliques over a graph which are the fundamental objects of the Hammersley Clifford's Theorem, the right side refer to identifying some filters over a tree which are the fundamental objects of Theorem 3.

Step 1: Identification of the invariance properties.

Definition 18 *Let χ be a coloration on the arborescence A. Then χ is said to have the blackening consistency property if*

$$\forall z \in A, \quad \chi(z) = \text{black} \quad \textit{if and only if} \quad \forall z_j \in C(z), \quad \chi(z_j) = \text{black}.$$

This rule propagates the blackening color between branch nodes and leaf nodes. Under this constraint, any node z_i depends on the set of nodes $\{Sib(z_i) \cup P(z_i)\}$. Moreover, if a node z_i does not depends on z_j (i.e. we can blacken z_j without changing the coloration of z_i) then it also does not depend on $C(z_j)$. More generally, the set $z_i \cup Sib(z_i) \cup A(z_i) \cup D(z_i)$ and its complementary are independent.

Definition 19 *Let χ be a coloration on the arborescence A. A random variable is said to be* blackening consistent *if it respects the following property:*

$$\forall X \subset Z, \quad \mathbb{P}\left(\chi^{X \cup D(X)} | \chi^{Z \setminus (X \cup D(X))}\right) = \mathbb{P}\left(\chi^{X \cup D(X)} | \chi^{Sib(X) \cup P(X)}\right). \tag{16}$$

Step 2: Construction of the associated blackening operators

We now want to construct the blackening operator associated to the blackening consistency property. Consider a coloration χ which satisfies the blackening consistency property. Let $z_j \in Z$. We analyze the invariance of $\log(\mathbb{P}(\chi^{z_j}))$ under some blackening operators.

Let z_i be another node of the arborescence. Using the analysis of Step 1, we have only two possibilities:

- Case 1: z_j is independent of z_i, the prediction does not depends on z_i and thus the log probability is invariant under the action of the operator $B_{z_i \cup D(z_i)}$.
$$B_{z_i \cup D(z_i)} \log(\mathbb{P}(\chi^{z_j}) = \log(\mathbb{P}(\chi^{z_j}),$$
which can be reformulated using the propositional function (see Definition 14) as
$$\psi_{\log(\mathbb{P}(\chi^{z_j})}(B_{z_i \cup D(z_i)}) = \text{True}.$$
- Case 2: z_j depends on z_i, thus $z_j \in \{z_i \cup A(z_i) \cup D(z_i) \cup Sib(z_i)\}$. As the set $\{z_i \cup A(z_i) \cup D(z_i) \cup Sib(z_i)\}$ is independent of its complementary, the log probability is invariant under the action of the operator $B_{Z \setminus (z_i \cup A(z_i) \cup D(z_i) \cup Sib(z_i))}$, i.e.
$$B_{Z \setminus (z_i \cup A(z_i) \cup D(z_i) \cup Sib(z_i))} \log(\mathbb{P}(\chi^{z_j}) = \log(\mathbb{P}(\chi^{z_j}),$$

which can be reformulated using the propositional function (see Definition 14) as

$$\psi_{\log(\mathbb{P}(\chi^{z_j})}(B_{Z\setminus(z_i\cup A(z_i)\cup D(z_i)\cup Sib(z_i))}) = \text{True}.$$

This two cases represent all the possibilities, thus

$$\psi_{\log(\mathbb{P}(\chi^{z_j})}(B_{z_i\cup D(z_i)}) \cup \psi_{\log(\mathbb{P}(\chi^{z_j})}(B_{Z\setminus(z_i\cup A(z_i)\cup D(z_i)\cup Sib(z_i))}) = \text{True}.$$

Hence, using the link between the union in propositional logic and the union in Boolean rings, we deduce that

$$\psi_{\log(\mathbb{P}(\chi^{z_j})}(B_{z_i\cup D(z_i)} \vee B_{Z\setminus(z_i\cup A(z_i)\cup D(z_i)\cup Sib(z_i))}) = \text{True}.$$

To simplify the notations, we can introduce β_{z_i} as

$$\beta_{z_i} = B_{Z\setminus(z_i\cup A(z_i)\cup D(z_i)\cup Sib(z_i))} \vee B_{z_i\cup D(z_i)}. \tag{17}$$

Then

$$\psi_{\log(\mathbb{P}(\chi^{z_j})}(\beta_{z_i}) = \text{True}.$$

As this result is true for all $z_i \in Z$, we have

$$\bigcap_{z_i\in Z} \psi_{\log(\mathbb{P}(\chi^{z_j})}(\beta_{z_i}) = \text{True}.$$

Using again the link between the intersection in propositional logic and the intersection in Boolean ring, we get

$$\psi_{\log(\mathbb{P}(\chi^{z_j})}(\prod_{z_i\in Z} \beta_{z_i}) = \text{True}.$$

To simplify the notations, we can introduce

$$\beta = \prod_{z_i\in Z} \beta_{z_i}. \tag{18}$$

Then

$$\psi_{\log(\mathbb{P}(\chi^{z_j})}(\beta) = \text{True}.$$

Step 3: Link with the probability function

This step, which states that $\log(P(\chi))$ is invariant under β, is purely calculative as the relation between probability and Blackening operators is already presented in Step 6.2. The full proof can be found in Appendix A.10.

Step 4: Reduction of the blackening operators

We can now formulate the factorization of β in the following lemma.

Lemma 7 *$I(\beta)$ can be decomposed as*

$$I(\beta) = \sum_{I\in F(G)} I(B_I), \tag{19}$$

where $F(G)$ is the set of all the sets of leaves with their common ancestors.

The Proof of Lemma 7 can be found in supplementary materials E.10. The main idea of the proof is to identify the structure which do not vanish under the conditions given in Corollary 1.

The proof of Theorem 3 is complete.

6.3 Experimentations

6.3.1 Dataset presentation

The dataset used in our study was introduced by [25]. This dataset stands out as one of the limited resources available that provides token and sentence-level annotations for argument units. It comprises 8,000 sentences, evenly distributed across 8 topics, and can be employed for the tasks of argument unit identification and argument polarity identification. Within each topic, the words in the sentences are annotated with three labels: PRO (supporting arguments), CON (opposing arguments), and NON (non-argumentative). We selected this dataset due to its stringent rules for propagating token-level labels to sentence-level labels. Specifically, if only NON labels are present, the sentence is labeled as NON.

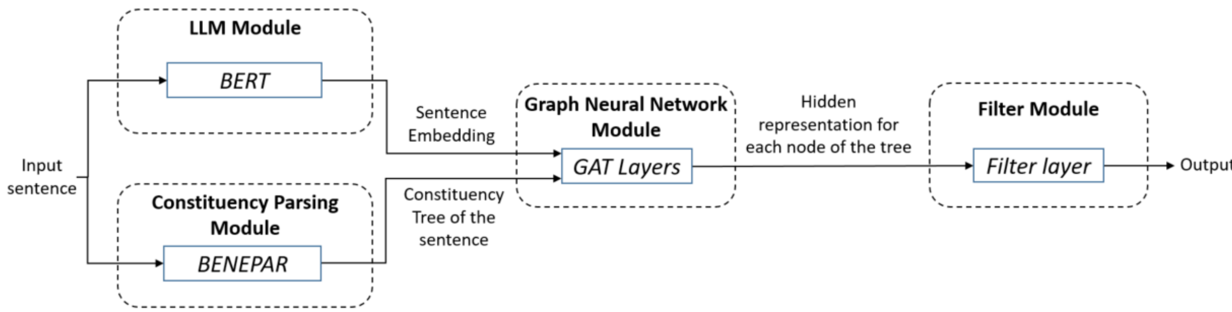

Figure 3: Illustration of the architecture of the proposed model incorporating a filter layer.

In cases where there are only PRO labels (or only CON labels), the corresponding label is assigned. When both PRO and CON labels are present, the more frequently occurring label is assigned (or randomly chosen in the event of a tie). The PRO and CON labels therefore prevail over the NON label.

We assigned labels to the internal nodes (NT) of the constituency tree to capture their representations. Since these annotations were not available in the dataset, we opted to annotate the internal nodes using the same labeling rules described earlier for sentence labeling. This decision ensures the internal logic of the labeling process is preserved.

6.3.2 Exploitation of our Theorem 3 in the construction of the model

By considering node labeling equivalent to node coloring, as discussed in Subsection 3.2, it can be seen that our annotation rules defined in the previous subsection adhere to the consistency property outlined in Definition 18. This motivates us to factorize the model with respect to the filter of the arborescence. Following Theorem 3, we are authorized to factorize the log probability of the labelization of the arborescence into a sum of components reliant solely on the filters present within the tree.

To assess the effectiveness of our approach compared to the baseline method proposed by [25], we replaced the CRF layer from their original paper with a combination of a GAT [26] layer and a novel filter layer. As presented in Figure 3, our model consists of an LLM (BERT [27]) for token embedding, followed by a GAT layer to obtain embeddings for interior nodes, and the results are further refined using the filter layer.

Our model architecture, as depicted in Figure 3, consists of four modules.

- **Step 1: Calculation of sentence embedding using a Language Model (LLM)**. To implement the BERT model, we used the transformers library [28]. This choice aligns with the original approach taken in the reference article [25], allowing us to compare our results with their baseline effectively. This module is called **"LLM Module"** in Figure 3.
- **Step 2: Construction of the sentence's constituent tree**. To construct the constituency tree, we employed the Berkeley Neural Parser (BENEPAR) [29]. BENEPAR is a multilingual constituency parser that benefits from unsupervised pre-training across multiple languages. This model provides weights for 12 languages and offers a simple API integrated into Spacy. Thus, it serves as an excellent choice for accurate results within a larger pipeline. However, this additional preprocessing step necessitates an exact match between the tokenizations used by BENEPAR and BERT models. Consequently, sentences that do not have a match were excluded, including samples from the in-domain test set. As a result, the cross-domain split had 3960 samples in the training set (instead of 4000), 790 samples in the development set (instead of 800), and 1959 samples in the test set (instead of 2000). Similarly, the in-domain split had 4157 samples in the training set (instead of 4200), 593 samples in the development set (instead of 600), and 1159 samples in the test set (instead of 1200). In total, less than 2 % of the dataset was removed. This module is called **"Constituency Parsing Module"** in Figure 3.
- **Step 3: Computation of a hidden representation for each word through a Graph Neural Network based on message passing**. The GNN module is implemented using the PyTorch Geometric library [30], specifically using the Graph Attention Network (GAT) model. During the development of this model, one of the challenges encountered was the management of batches. In PyTorch Geometric, each input is associated with a graph represented by a two-dimensional matrix containing edges between nodes. To handle graphs of varying sizes, the batch matrix used in PyTorch is unfolded into a list, and a list of indices is used to link the input data to the batch indices. Consequently, switching between these two representations to incorporate both BERT and

GNNs within our model induces a computational speed reduction. This module is called **"Graph Neural Network Module"** in Figure 3.

- **Step 4: Calculating the final output with Recurrent Neural Networks on Filter Structures**. Upon completing these three steps, we leverage the outcome of Theorem 3 to construct the final module of our model. Since the node count of each filter varies, we opted for multiple RNN layers. We input the nodes composing the filter in a top-to-bottom order based on the tree structure. The multi-layer RNN module, is designed to run on each individual tree filter. Its purpose is to facilitate information propagation exclusively among nodes residing within the same filter. Employing an RNN for this task could have been substituted with alternatives like an LSTM or an attention layer. This module is called **"Filter Module"** in Figure 3.

6.3.3 Training and hyperparameters optimization

To select the optimal hyperparameters for our model, we have chosen the Optuna library [31], which is known for its cost-effective hyperparameter optimization capabilities. Optuna offers two key advantages for our study.

Searching strategy: Optuna employs a relational sampling method that can uncover underlying relationships among hyperparameters through independent samplings.

Pruning strategy: Optuna incorporates the Asynchronous Successive Halving algorithm [32] to interrupt unpromising trials based on intermediate F1 score values. This algorithm allows trials to continue only if their actual F1 scores are among the best intermediate results.

In the overall training process, we loaded the BERT model and a data batch into memory. Our implementation required 30 GB of memory (GPU or CPU). Consequently, we conducted the global training experiments on the CPU, which took approximately 2 hours per iteration, resulting in around 30 iterations per model configuration. On the other hand, the transfer learning model training consumed about 7 GB of memory, allowing it to run on the GPU. Hence, we performed approximately 200 trials for model hyperparameter optimization.

Table 2 provides an assessment of the significance of different hyperparameters in our model. Notably, the learning rate and the maximum allowed gradient value emerged as the most crucial hyperparameters. Empirically, we observed that when the gradient is unconstrained, the model tends to converge to a local optimum where each word is assigned the label "NO." This local optimum arises due to the dataset's imbalance, which predominantly favors the absence of an argument.

	Test Intervals	Best values	Parameters importance
Learning rate	10^{-5} to 10^{-3}	$2.8 \cdot 10^{-5}$	30 %
Maximum gradient allowed	10^{-1} to 10^{2}	9.7	49 %
Number of GAT layers	1 to 3	2	2 %
Number of unit per GAT layers	50 to 300	290 and 100	2 %
Number of heads per GAT layers	1 to 3	3 and 3	7 %
Number of linear layers	1 to 3	2	5 %
Number of unit per linear layers	50 to 250	100 and 100	5 %

Table 2: Feature importance of the BERT-GAT-FILTER model

6.3.4 Results analysis

As shown in Table 3, our model achieves superior results for sentence-level prediction, aligning with the inherent structure of our model. The two main errors previously observed on this dataset were the span of an argumentative segments was not correctly recognized and the stances are not correctly classified. In this paper, we focus on the span detection problem and improve the method for identifying the boundaries of an ADU. However, we do not have a strict adherence to grammatical correctness. As mentioned in [25], this would require that all spans containing arguments be clauses.

6.3.5 Analyzing computational costs

The complexity of our approach heavily hinges on the nature of the studied dependency relation. Identifying filters within the tree structure, as depicted in Figure 2, holds low computational complexity. Filters in a tree consist of a set of leaf nodes sharing a common parent node with all ancestors up to the root. For instance, in a bipartite graph, the complexity scales by $1.5 * m$, with m representing the number of leaf nodes.

	Token level	Sentence Level
Majority baseline	0.254 *	0.211 *
FLAIR	0.613 *	0.620 *
BERT base	0.654 *	0.673 *
BERT large	0.683 *	0.709 *
BERT large - Linear chain CRF	0.696 *	0.744 *
BERT large - GAT - Filter model	**0.72**	**0.76**
Human performance	0.763 *	0.799 *

Table 3: F1- score of the different models at token level and sentence level on the test dataset (the results with a star (*) have been directly taken from [25]), they are reproduced in the code presents in supplementary material.

In contrast, as highlighted by the Hammersley-Clifford theorem and exemplified by Conditional Random Fields, the pursuit of cliques in graphs assumes a notably intricate character. In essence, this intricacy stems from the fact that the maximum clique problem aligns itself within Karp's catalog of 21 NP-complete challenges [33]. Notably, in the domain of natural language processing (NLP), the application is often streamlined to view sentences as linear chains of words [15].

Furthermore, akin to the formula employed in conditional random fields, our methodology furnishes a formula for factorizing the logarithm of the probability. This grants considerable flexibility in selecting the implementation of the function S as outlined in Theorem 3. In our specific example, we have chosen a formulation that involves averaging the embeddings of the nodes within the filter using Recurrent Neural Networks (RNNs). However, we can also investigate more efficient methods for this purpose.

7 Conclusion and future works

In this paper, we generalized the Hammersley-Clifford theorem [5] to principal ideals on Boolean rings. This allowed to identify relations between blackening operators and Boolean algebra. We then proposed a new method to analyse data structure and nodes relationship. Finally, we illustrated this method on a specific tree structure. This demonstrate the efficiency of our method when using domain assumption of our data which then allow to deduce the underlying invariant structure.

We are confident that the framework, referred to as the "strategy" in Subsection 5.1, holds substantial promise for unveiling new factorization structures within graph dependencies. Notably, these structures can transcend the conventional neighborhood relationships relied upon in the Hammersley-Clifford theorem. To provide greater clarity, our focus lies on non-symmetric relationships. This distinction becomes particularly relevant in scenarios where symmetric dependency relationships govern graph nodes. In such instances, a transformative approach can be employed to represent the dependency relationship as edges in a new graph, effectively aligning it with the principles underlying the classical Hammersley-Clifford theorem.

In future work, we will delve into the full utilization of this framework, aiming to harness sparsity techniques and Boolean logic gates to learn and leverage the invariant structure.

A Appendix

A.1 Equivalence between Boolean algebras and Boolean rings

In this Section, we introduce more precisely the notion of Boolean algebra, a mathematical structure that is isomorphic to Boolean rings but defined using the meet $\wedge$ and join $\vee$ operators instead of the $\oplus$ and $\otimes$ operators. We use the definition from [11] based on partially orders sets and lattices, which is equivalent to the definition based on $\vee$ and $\wedge$ presented in [24] and used by [34].

To begin with, let us recall a few classical definitions. Let $(\mathcal{O}, \leq)$ be a partially ordered set and $J \subset \mathcal{O}$. An element $a \in \mathcal{O}$ is an *upper bound* (resp. a *lower bound*) of J if, for all $b \in J$, we have $b \leq a$ (resp. $a \leq b$). An upper bound (resp. a lower bound) $a \in \mathcal{O}$ of J is said to be a *least upper bound* (resp. a *greatest lower bound*) of J if every upper bound (resp. *lower bound*) c of J satisfies $a \leq c$ (resp. $c \leq a$).

Definition 20 *A partially order set $\mathbb{L}$ is called a* lattice *if every pair $(x, y) \in \mathbb{L}^2$ has a least upper bound and a greatest lower bound, respectively denoted by $x \vee y$ and $x \wedge y$. Moreover,*

1. *the lattice $\mathbb{L}$ is said to be* distributive *if, for all $(x, y, z) \in \mathbb{L}^3$, we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;*
2. *the lattice is said to have a unit if there exists a unique $\mathbb{1} \in \mathbb{L}$ such that $x \leq \mathbb{1}$ for all $x \in L$;*
3. *the lattice is said to have a zero if there exists a unique element $0 \in \mathbb{L}$ such that $0 \leq x$ for all $x \in \mathbb{L}$;*
4. *the lattice is complemented if it has a unit and a zero and if, for every $x \in \mathbb{L}$, there is an element $x' \in \mathbb{L}$ (called the complement of x) such that $x \wedge x' = 0$ and $x \vee x' = \mathbb{1}$.*

Definition 21 *A distributive and complemented lattice is called a* Boolean algebra*.*

With the notations of Definition 20, we can identify a Boolean algebra by a quintuple:

$$(\mathbb{L}, \vee, \wedge, 0, \mathbb{1}).$$

The following proposition (whose proof can be found in [34]) makes a link between Boolean rings and Boolean algebras.

Proposition 3 *Let B be a Boolean algebra equipped with the operators $\vee$ and $\wedge$. Then B can be converted into a Boolean ring with respect to the addition $\oplus$ and the multiplication $\otimes$ defined by*

$$a \oplus b = (a \wedge b') \vee (a' \wedge b) \quad \textit{and} \quad a \otimes b = a \wedge b.$$

Conversely, a Boolean ring with partial order defined by Definition 1 Section 3 is a Boolean algebra and we have

$$a \vee b = a \oplus b \oplus a \otimes b \quad \textit{and} \quad a \wedge b = a \otimes b.$$

Hence, one can consider the three operations $\oplus$, $\wedge$ (equivalent to $\otimes$) and $\vee$ as three operations operating on a Boolean ring.

A.2 Proof of Lemma 2

Proof 3 (Proof of Lemma 2) *Let W, X, Y be three subsets of Z, let F be a function of $\mathcal{F}$ and χ be a coloration.*

- *Complementarity.*

  We will first prove that $\mathbb{1} - B_X$ is a solution of Equation 6. We have indeed

$$\begin{aligned} B_X \wedge (\mathbb{1} - B_X) &= B_X - B_X \wedge B_X \\ &= B_X - B_X \\ &= 0 \end{aligned}$$

  and

$$\begin{aligned} B_X \vee (\mathbb{1} - B_X) &= B_X + (\mathbb{1} - B_X) - B_X \wedge (\mathbb{1} - B_X) \\ &= B_X + (\mathbb{1} - B_X) - 0 \\ &= \mathbb{1}. \end{aligned}$$

  Let us show that it is the unique solution of Equation 6.

  Let B be a solution of Equation 6. Then, we have

$$B_X \wedge B = 0 \quad \textit{and} \quad B_X \vee B = B_X + B - B_X \wedge B = \mathbb{1}.$$

  By inserting the first equation into the second, we obtain the system

$$B_X \wedge B = 0 \quad \textit{and} \quad B_X + B = \mathbb{1}.$$

  Therefore $B = \mathbb{1} - B_X$.

  Let us prove the two De Morgan's laws. We have

$$\begin{aligned} \neg(B_X \vee B_Y) &= \mathbb{1} - B_X \vee B_Y \\ &= \mathbb{1} - (B_X + B_Y - B_X \wedge B_Y) \\ &= (\mathbb{1} - B_X) - B_Y \wedge (\mathbb{1} - B_X) \\ &= (\mathbb{1} - B_X) \wedge (\mathbb{1} - B_Y) \\ &= \neg B_X \wedge \neg B_Y \end{aligned}$$

and

$$\begin{aligned}\neg(B_X \wedge B_Y) &= \mathbb{1} - B_X \wedge B_Y \\ &= \mathbb{1} + (\mathbb{1} - \mathbb{1}) + (B_X - B_X) + (B_Y - B_Y) - B_X \wedge B_Y \\ &= (\mathbb{1} - B_X) + (\mathbb{1} - B_Y) - (\mathbb{1} - B_X - B_Y + B_X \wedge B_Y) \\ &= (\mathbb{1} - B_X) + (\mathbb{1} - B_Y) - (\mathbb{1} - B_X) \wedge (\mathbb{1} - B_Y) \\ &= \neg B_X \vee \neg B_Y.\end{aligned}$$

- *Commutativity*
  - *We first prove the commutativity of the operator $\wedge$:*

$$\begin{aligned}(B_W \wedge B_X)F(\chi) &= B_W F(\chi_X) = F(\chi_{X\cup W}) \\ &= B_X F(\chi_W) = (B_X \wedge B_W)F(\chi)\end{aligned}$$

and

$$\begin{aligned}B_W \wedge \neg B_X &= B_W \wedge (\mathbb{1} - B_X) = B_W - B_W \wedge B_X \\ &= B_W - B_X \wedge B_W = (\neg B_X \wedge B_W).\end{aligned}$$

  - *We now prove the commutativity of the operator $\vee$:*

$$\begin{aligned}B_W \vee B_X &= B_W + B_X - B_W \wedge B_X = B_X + B_W - B_W \wedge B_X \\ &= B_X + B_W - B_X \wedge B_W = B_X \vee B_W\end{aligned}$$

and

$$\begin{aligned}B_W \vee \neg B_X &= B_W + \neg B_X - B_W \wedge \neg B_X = \neg B_X + B_W - B_W \wedge \neg B_X \\ &= \neg B_X + B_W - \neg B_X \wedge B_W = \neg B_X \vee B_W.\end{aligned}$$

- *Associativity*
  - *We first prove the associativity of the operator $\wedge$:*

$$\begin{aligned}((B_W \wedge B_X) \wedge B_Y)F(\chi) &= (B_W \wedge B_X)F(\chi_Y) = F(\chi_{W\cup X\cup Y}) \\ &= B_W \wedge F(\chi_{X\cup Y}) = B_W \wedge (B_X \wedge B_Y) \circ F(\chi).\end{aligned}$$

  - *We now prove the associativity of the operator $\vee$:*

$$\begin{aligned}(B_W \vee B_X) \vee B_Y &= (B_W \vee B_X) + B_Y - (B_W \vee B_X) \wedge B_Y \\ &= (B_W + B_X - B_{W\cup X}) + B_Y - (B_W + B_X - B_{W\cup X}) \wedge B_Y \\ &= B_W + B_X + B_Y - B_{W\cup X} - B_{W\cup Y} - B_{X\cup Y} + B_{W\cup X\cup Y} \\ &= (B_X + B_Y - B_{X\cup Y}) + B_W - B_W \wedge (B_X + B_Y - B_{X\cup Y}) \\ &= B_W \vee (B_X \vee B_Y).\end{aligned}$$

- *Distributivity*
  - *We prove the distributivity the $\wedge$ over $\vee$:*

$$\begin{aligned}B_W \wedge (B_X \vee B_Y) &= B_W \wedge (B_X + B_Y - B_{X\cup Y}) \\ &= B_W \wedge B_X + B_W \wedge B_Y - B_{W\cup X\cup Y} \\ &= B_W \wedge B_X + B_W \wedge B_Y - B_{W\cup X} \wedge B_{W\cup Y} \\ &= (B_W \wedge B_X) \vee (B_W \wedge B_Y).\end{aligned}$$

A.3 Proof of Lemma 3

Proof 4 (Proof of Lemma 3) *1. Let P be a polynomial operator. By Lemma 2,*

$$\begin{aligned}\neg P &= \neg(\bigvee_{1\le a\le l} M_{X_a,Y_a}) \\ &= \bigwedge_{1\le a\le l}(\neg M_{X_a,Y_a}) \\ &= \bigwedge_{1\le a\le l}(\neg(\bigwedge_{1\le i\le n} B_{X_{a,i}} \bigwedge_{1\le j\le m}(\neg B_{Y_{a,j}}))) \\ &= \bigwedge_{1\le a\le l}(\bigvee_{1\le i\le n}(\neg B_{X_{a,i}}) \bigvee_{1\le j\le m} B_{Y_{a,j}}).\end{aligned}$$

This formulation can be rewritten as a sum of products because of the distributivity of $\wedge$ over $\vee$. Thus, $\neg P$ is a polynomial operator.

The stability by $\vee$ is obvious and the stability by $\wedge$ comes from the distributivity of $\wedge$ over $\vee$.

2. *The fact that the polynomial operators commute comes again from the distributivity of $\wedge$ over $\vee$ and the fact that pure blackening operators commute.*

3. *Let P and Q be two elements of $\mathcal{P}$ which are projectors. Since P and Q commute, we have*

$$(P \wedge Q)^2 = (P \wedge Q) \wedge (P \wedge Q) = (P \wedge P) \wedge (Q \wedge Q) = P \wedge Q,$$

$$\begin{aligned}(P \vee Q)^2 &= (P \vee Q) \wedge (P \vee Q)\\ &= (P + Q - P \wedge Q) \wedge (P + Q - P \wedge Q)\\ &= (P^2 + P \wedge Q - P \wedge Q) + (P \wedge Q + Q^2 - P \wedge Q) - (P \wedge Q)^2\\ &= P + Q - P \wedge Q = P \vee Q,\end{aligned}$$

$$(\neg P)^2 = (\mathbb{1} - P) \wedge (\mathbb{1} - P), = (\mathbb{1} - P) - P + P^2 = \mathbb{1} - P = \neg P.$$

Now we notice that pure blackening operators are projectors. Hence, by direct induction, monomial blackening operators are projectors, then polynomial blackening operator are also projectors.

A.4 Proof of Lemma 4

Proof 5 (Proof of Lemma 4) *Let P be an element of $\mathcal{P}$, by Item 3 of Lemma 3,*

$$P \wedge P = P.$$

Thus, $P \leq_{\mathcal{P}} P$ and $\leq_{\mathcal{P}}$ is reflexive.

Let P and Q be two elements of $\mathcal{P}$ such that $P \leq_{\mathcal{P}} Q$ and $Q \leq_{\mathcal{P}} P$. Then, we have

$$P \wedge Q = P \quad \text{and} \quad Q \wedge P = Q.$$

Item 2 of Lemma 3 implies that $P = Q$. Therefore $\leq_{\mathcal{P}}$ is anti-symmetric.

Let P, Q and R be three elements of $\mathcal{P}$ such that $P \leq_{\mathcal{P}} Q$ and $Q \leq_{\mathcal{P}} R$. Then,

$$P \wedge R = P \wedge (Q \wedge R) = (P \wedge Q) \wedge R = Q \wedge R = R.$$

This proves that $P \leq_{\mathcal{P}} R$ and that $\leq_{\mathcal{P}}$ is transitive.

We have proved that $\leq_{\mathcal{P}}$ is a partial order. We will now prove that $P \vee Q$ is the greatest lower bound of the set $\{P, Q\}$. Let P, Q be two elements of $\mathcal{P}$, we have

$$(P \vee Q) \wedge P = P \wedge P + P \wedge Q - P \wedge Q \wedge P = P$$

and

$$(P \vee Q) \wedge Q = P \wedge Q + Q \wedge Q - P \wedge Q \wedge Q = Q.$$

Thus $P \vee Q \leq_{\mathcal{P}} P$ and $P \vee Q \leq_{\mathcal{P}} Q$. This means that $P \vee Q$ is a lower bound of the set $\{P, Q\}$.

Let R such that $R \leq_{\mathcal{P}} P$ and $R \leq_{\mathcal{P}} Q$. Then, we have

$$R \wedge (P \vee Q) = (R \wedge P) + (R \wedge Q) - (R \wedge P \wedge Q) = P + Q - P \wedge Q = P \vee Q.$$

Thus, $R \leq_{\mathcal{P}} (P \vee Q)$ and $P \vee Q$ is the greatest lower bound of the set $\{P, Q\}$. We will now prove that $P \wedge Q$ is the least upper bound of the set $\{P, Q\}$.

Let P, Q and R be three elements of $\mathcal{P}$ such that $P \leq_{\mathcal{P}} R$ and $Q \leq_{\mathcal{P}} R$. We have

$$(P \wedge Q) \wedge P = P \wedge Q \quad \text{and} \quad (P \wedge Q) \wedge Q = P \wedge Q.$$

Thus, $P \wedge Q$ is an upper bound of the set $\{P, Q\}$.

Then, we have

$$(P \wedge Q) \wedge R = P \wedge (Q \wedge R) = P \wedge R = R.$$

Thus $(P \wedge Q) \leq_{\mathcal{P}} R$ and $P \wedge Q$ is the least upper bound of the set $\{P, Q\}$.

Finally, the fact that $\neg$ is the complementary operator is direct by definition of $\neg$ and the fact that $\vee$ and $\wedge$ are the least upper bound and the greatest lower bound.

A.5 Proof of Proposition 1

Proof 6 (Proof of Proposition 1) *As a straightforward consequence of Lemma 2, the lattice $(\mathcal{P}, \leq)$ is distributive and every element has a unique complement.*

Let P be an operator on $\mathcal{F}$. Then,

$$(PB_Z)F(\chi) = PF(\chi_Z) = 0$$
$$(PB_\emptyset)F(\chi) = PF(\chi).$$

Thus, B_Z (resp. $B_\emptyset$)is the neutral element for $\vee$ (resp. $\wedge$) on the set of the operators on $\mathcal{F}$. They are in particular the neutral elements of $(\mathcal{P}, \leq)$.

We have proved that $(\mathcal{P}, \leq)$ is complemented. It is a Boolean algebra, which concludes the proof of Proposition 1.

A.6 Proof of Corollary 1

Proof 7 (Proof of Corollary 1) *In order to simplify the notation, we introduce Γ as:*

$$\Gamma = \left(\bigcap_{k_1 \in K} I(a_{k_1}) \bigcap_{k_2 \in J \setminus K} I(b_{k_2}) \bigcap_{k_3 \in J \setminus K} I(a'_{k_3}) \right)$$

In order that $\Gamma \neq \{0\}$, every sub-product of ideals composing Γ must be different from the empty set. Thus, we can deduce some necessary conditions by extracting some interesting sub-products.

1. $\bigcap_{k_1 \in K} I(a_{k_1}) \neq \{0\}$ *and* $\bigcap_{k_2 \in J \setminus K} I(b_{k_2}) \neq \{0\}$
2. $\forall k_3 \in J \setminus K, \quad I(a'_{k_3}) \cap \bigcap_{k_2 \in J \setminus K} I(b_{k_2}) \neq \{0\}$
3. $\forall k_3 \in J \setminus K, \quad I(a'_{k_3}) \cap \bigcap_{k_1 \in K} I(a_{k_1}) \neq \{0\}$
4. $\forall k_1 \in K, \forall (k_2) \in J \setminus K, \quad I(a_{k_1}) I(b_{k_2}) \neq \{0\}$
5. $\forall (k_3, k'_3) \in (J \setminus K)^2, \quad I(a'_{k_3}) \cap I(a'_{k'_3}) \neq \{0\}$

Using the property of Lemma 1 and replacing $a_i = B_{X_i}$, $b_i = B_{Y_i}$ and $\{0\} = I(B_Z)$. The system of necessary conditions in order that Γ is not the empty set can be reformulated as:

1. $B_{\bigcup_{k \in K} X_k} \neq B_Z$ *and* $B_{\bigcup_{j \in (J \setminus K)} Y_j} \neq B_Z$.
2. $\forall j \in (J \setminus K), \quad X_j \not\subset \bigcup_{j' \in (J \setminus K)} Y_{j'}$.
3. $\forall j \in J \setminus K, \forall k \subset K, \quad X_j \not\subset X_k$.
4. $\forall j \in J \setminus K, \forall k \subset K, \quad B_{X_k} \cap B_{Y_j} \neq B_Z$.
5. $\forall (j, j') \in (J \setminus K)^2, \quad (\mathbb{1} - B_{X_j})(\mathbb{1} - B_{X_{j'}}) \neq 0$.

which yields the result.

A.7 Proof of Lemma 5

Proof 8 (Proof of Lemma 5) *We can notice that $I(\beta)$ has the same form as the ideal studied in Corollary 1 with $\forall i \in J, X_i = z_i$ and $Y_i = Z\backslash(z_i \cup \partial z_i)$. The system of necessary conditions in order that Γ is not null can be reformulated as:*

1. $\bigcup_{k\in K} z_k \neq Z$ *and* $\bigcup_{j\in(J\backslash K)} Z\backslash(z_i \cup \partial z_i) \neq Z$. *i.e.* $\bigcap_{j\in(J\backslash K)}(z_i \cup \partial z_i) \neq \emptyset$
2. $\forall j \in (J \setminus K)$, $X_j \not\subset \bigcup_{j'\in(J\backslash K)} Y_{j'}$.
3. $\forall k_1 \in K$, $\forall k_3 \in J \setminus K$, $\{z_{k_3}\} \not\subset \{z_{k_1}\}$ *i.e.* $z_{k_3} \neq z_{k_1}$,
4. $\forall k_1 \in K$, $\forall k_2 \in J \setminus K$, $z_{k_1} \cup Z\backslash(z_{k_2} \cup \partial z_{k_2}) \neq Z$, *i.e.* $(z_{k_2} \cup \partial z_{k_2}) \neq z_{k_1}$,
5. $\forall (k_2, k_3) \in J \setminus K$, $z_{k_3} \not\subset Z\backslash(z_{k_2} \cup \partial z_{k_2})$ *i.e.* $z_{k_3} \subset z_{k_2} \cup \partial z_{k_2}$.

The condition 1 is directly satisfied if K does not contain all the nodes and their exist some nodes which are not neighbour of nodes in $Z\backslash K$.

The condition 3 is always verified because $K \cap (J\backslash K) = \emptyset$.

The condition 5 implies that every element in $J\backslash K$ are all neighbours two by two. Which means that $J\backslash K$ is a clique. Thus, $J\backslash K$ is a clique is a necessary condition in order that the element is not null.

We now suppose that, $J\backslash K$ is a clique. Then, the condition 2 and 5 are verified.

Thus, we can reduce the formula of $I(\beta)$ to the sum over the cliques of the graph:

$$I(\beta) = \sum_{X\in L(Z)} I(B_{Z\backslash X}) = I(\sum_{X\in L(Z)} B_{Z\backslash X}).$$

A.8 Proof of Lemma 6

Proof 9 (Proof of Lemma 6) *First, we can notice that when we are predicting χ^X, blackening other nodes than X have no effect. Thus, we have the equality*

$$\forall X \subset Z, \forall Y \quad \text{such that} \quad X \wedge Y = \emptyset, \quad \mathbb{P}(\chi_Y^X) = \mathbb{P}(\chi^X). \tag{20}$$

Let $X \subset Z$. By using the Markovian assumption, we get

$$\begin{aligned} Q_Z &= \log(\mathbb{P}(\chi^Z)) = \log(\mathbb{P}(\chi^X, \chi^{Z\backslash X})) \\ &= \log(\mathbb{P}(\chi^{Z\backslash X})\mathbb{P}(\chi^X|\chi^{Z\backslash X})) = \log(\mathbb{P}(\chi^{Z\backslash X})\mathbb{P}(\chi^X|\chi^{\partial X})) \\ &= \log(\mathbb{P}(\chi^{Z\backslash X})\mathbb{P}(\chi^X, \chi^{\partial X})/\mathbb{P}(\chi^{\partial X})) = \log(\mathbb{P}(\chi^{Z\backslash X})) + \log(\mathbb{P}(\chi^{X\cup\partial X})) - \log(\mathbb{P}(\chi^{\partial X})) \\ &= Q_{Z\backslash X} + Q_{X\cup\partial X} - Q_{\partial X}. \end{aligned} \tag{21}$$

Let Y be another subset of Z. Applying Equation 20 to Equation 21 yields

$$B_Y Q_Z = B_Y Q_{Z\backslash X} + B_Y Q_{X\cup\partial X} - B_Y Q_{\partial X},$$

which is equivalent to

$$\frac{\mathbb{P}(\chi_Y)}{\mathbb{P}(\chi_Y^{X\cup\partial X)})} = \frac{\mathbb{P}(\chi_Y^{Z\backslash X})}{\mathbb{P}(\chi_Y^{\partial X})}.$$

Let us now prove that

$$Q_{X\cup\partial X} - B_X Q_{X\cup\partial X} = B_{Z\backslash\partial X} Q_Z - B_{Z\backslash(X\cup\partial X)} Q_Z. \tag{22}$$

To this aim, we denote

$$S = \frac{\mathbb{P}(\chi^{X\cup\partial X})}{\mathbb{P}(\chi_X^{X\cup\partial X})} - \frac{\mathbb{P}(\chi_{Z\backslash(X\cup\partial X)})}{\mathbb{P}(\chi_{Z\backslash\partial X})}.$$

From Equation 20 we get

$$\mathbb{P}(\chi^{X\cup\partial X}) = \mathbb{P}(\chi^{X\cup\partial X}_{Z\backslash(X\cup\partial X)})$$

and

$$\mathbb{P}(\chi_X^{X\cup\partial X}) = \mathbb{P}(\chi_{Z\setminus\partial X}^{X\cup\partial X}),$$

which leads to

$$S = \frac{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{X\cup\partial X})}{\mathbb{P}(\chi_{Z\setminus\partial X}^{X\cup\partial X})} - \frac{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)})}{\mathbb{P}(\chi_{Z\setminus\partial X})}.$$

Then,

$$S\frac{\mathbb{P}(\chi_{Z\setminus\partial X})}{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{X\cup\partial X})} = \frac{\mathbb{P}(\chi_{Z\setminus\partial X})}{\mathbb{P}(\chi_{Z\setminus\partial X}^{X\cup\partial X})} - \frac{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)})}{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{X\cup\partial X})}.$$

Using now Equation 21, with $Y = Z\setminus\partial X$

$$\frac{\mathbb{P}(\chi_{Z\setminus\partial X})}{\mathbb{P}(\chi_{Z\setminus\partial X}^{X\cup\partial X})} = \frac{\mathbb{P}(\chi_{Z\setminus\partial X}^{Z\setminus X})}{\mathbb{P}(\chi_{Z\setminus\partial X}^{\partial X})} = \frac{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{Z\setminus X})}{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{\partial X})}.$$

Using again Equation 21, with $Y = Z\setminus(X\cup\partial X)$, gives

$$\frac{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)})}{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{X\cup\partial X})} = \frac{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{Z\setminus X})}{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{\partial X})}.$$

Consequently, we have

$$S\frac{\mathbb{P}(\chi_{Z\setminus\partial X})}{\mathbb{P}(\chi_{Z\setminus(X\cup\partial X)}^{X\cup\partial X})} = 0,$$

and S = 0 which proves Equation 22.

Applying now B_X to Equation 21,

$$\begin{aligned}
B_XQ_Z &= B_XQ_{Z\setminus X} + B_XQ_{X\cup\partial X} - B_XQ_{\partial X}\\
&= B_X\log(\mathbb{P}(\chi^{Z\setminus X})) + B_XQ_{X\cup\partial X} - B_X\log(\mathbb{P}(\chi^{\partial X}))\\
&= \log(\mathbb{P}(\chi_X^{Z\setminus X})) + B_XQ_{X\cup\partial X} - \log(\mathbb{P}(\chi_X^{\partial X}))\\
&= \log(\mathbb{P}(\chi^{Z\setminus X})) + B_XQ_{X\cup\partial X} - \log(\mathbb{P}(\chi^{\partial X}))\\
&= Q_{Z\setminus X} + B_XQ_{X\cup\partial X} - Q_{\partial X}.
\end{aligned} \tag{23}$$

Now substracting Equation 21 to Equation 23, and using Equation 20

$$\begin{aligned}
Q_Z(\mathbb{1} - B_X) &= B_XQ_{X\cup\partial X} - Q_{X\cup\partial X}\\
&= B_{Z\setminus(X\cup\partial X)}Q_Z - B_{Z\setminus\partial X}Q_Z.
\end{aligned}$$

This leads to

$$Q_Z = B_XQ_Z + B_{Z\setminus(X\cup\partial X)}Q_Z - B_{Z\setminus\partial X}Q_Z = \beta_XQ_Z,$$

which end the proof.

A.9 Proof of Proposition 2

We first prove a useful lemma.

Lemma 8 *Let $P \in \mathcal{P}$, then for all $F \in \mathcal{F}$, we have*

$$\psi_F(P) = \text{True} \quad \textit{if, and only if,} \quad \mathbb{1} \in I(P). \tag{24}$$

Proof 10 (Proof of Lemma 8) *Let $P \in \mathcal{P}$, the notation $\mathbb{1} \in I(P)$ means that*

$$P \circ \mathbb{1} = \mathbb{1}.$$

This equality has to be understood in the sense of operator on $\mathcal{F}$, this is thus equivalent to

$$\forall F \in \mathcal{F}, \quad (P \circ \mathbb{1})F = \mathbb{1}F.$$

Thus

$$PF = F \quad \textit{which is equivalent to} \quad \psi_F(P) = \text{True}.$$

This ends the proof.

Proof 11 (Proof of Proposition 2) *We first prove Item 1 and Item 2. For all $F \in \mathcal{F}$, we have $B_\emptyset \circ F = F$ and $B_Z \circ F = 0$, so clearly $\psi_F(B_\emptyset) = \mathrm{True}$ and $\psi_F(B_Z) = \mathrm{False}$.*

Proof of Item 3 Suppose that $\psi_F(P) = \mathrm{True}$ then,

$$\begin{aligned}(P \vee Q)F &= PF + QF - (P \circ Q)F \\ &= F + QF - QF \\ &= F.\end{aligned}$$

As it is the same if $\psi_F(Q) = \mathrm{True}$, we have

$$\psi_F(P) \cup \psi_F(Q) = \mathrm{True} \quad \textit{implies that} \quad \psi_F(P \vee Q) = \mathrm{True}.$$

Suppose that

$$\psi_F(P \vee Q) = \mathrm{True}.$$

By using Lemma 8, we get

$$\mathbb{1} \in I(P \vee Q) = I(P) \cup I(Q).$$

Then it means, in terms of ideals,

$$\mathbb{1} \in I(P) \quad \textit{or} \quad \mathbb{1} \in I(Q).$$

Therefore, again by Lemma 8,

$$\psi_F(P) = \mathrm{True} \quad \textit{or} \quad \psi_F(Q) = \mathrm{True},$$

which leads to

$$\psi_F(P \vee Q) = \mathrm{True} \quad \textit{if and only if} \quad \psi_F(P) \cup \psi_F(Q) = \mathrm{True},$$

which ends the proof of Item 3.

Proof of Item 4. Suppose that

$$\psi_F(P) \cap \psi_F(Q) = \mathrm{True}.$$

Then

$$\psi_F(P) = \mathrm{True} \quad \textit{and} \quad \psi_F(Q) = \mathrm{True}$$

and

$$(P \wedge Q)F = P(QF) = PF = F.$$

We have proved that

$$\psi_F(P) \cap \psi_F(Q) = \mathrm{True} \quad \textit{implies that} \quad \psi_F(P \wedge Q) = \mathrm{True},$$

Suppose now that

$$\psi_F(P \wedge Q) = \mathrm{True},$$

then by using Lemma 8,

$$\mathbb{1} \in I(P \wedge Q) = I(P) \cap I(Q).$$

Therefore,

$$\mathbb{1} \in I(P) \quad \textit{and} \quad \mathbb{1} \in I(Q)$$

which means that

$$\psi_F(P) = \mathrm{True} \quad \textit{and} \quad \psi_F(Q) = \mathrm{True}.$$

We have proved that

$$\psi_F(P \wedge Q) = \mathrm{True} \quad \textit{if and only if} \quad \psi_F(P) \cap \psi_F(Q) = \mathrm{True},$$

which ends the proof of Item 4.

Item 5. *Suppose that*

$$\psi_F(\neg P) = \mathrm{True},$$

then by using Lemma 8 on $\neg P$,

$$(\mathbb{1} - P)F = F, \quad PF = 0.$$

Thus,

$$\psi_F(P) = \mathrm{False}.$$

Suppose that

$$\psi_F(P) = \text{True}$$

then by using Lemma 8,

$$PF = F, \quad (\mathbb{1} - P)F = 0.$$

Finally,

$$\psi_F(\neg P) = \text{False}.$$

We have thus proved that

$$\psi_F(\neg P) = \text{False} \quad \textit{if and only if} \quad \psi_F(P) = \text{True},$$

which is equivalent to

$$\psi_F(\neg P) = \neg\psi_F(P).$$

A.10 Proof of Step 6.2

Let T be an arboresence and χ be a coloration on the arboresence T. Let ω be a random coloring of Z which follows the filter invariant properties. Suppose that $\mathbb{P}(\omega = \chi_Z) \neq 0$. Let us state a useful lemma, whose proof is deferred to the end of this section.

Lemma 9 *Let χ be a coloration of the set of nodes Z and suppose that ω follows the filter invariant properties. We introduce the function*

$$\forall X \subset Z, \quad Q_X = \log(\mathbb{P}(\chi^X)).$$

Then, we have

$$\forall X \subset Z, \quad Q_Z = \beta_X Q_Z,$$

where β is defined in Section 4 Step 2.

Proof 12 (Proof of Step 6.2) *Let $X \subset Z$, using the filter invariant properties of ω, we get*

$$\begin{aligned}
Q_Z &= \log(P(\chi^Z)) = \log(P(\chi^{X \cup D(X)}, \chi^{Z \setminus \{X \cup D(X)\}})) \\
&= \log(P(\chi^{Z \setminus \{X \cup D(X)\}})P(\chi^{X \cup D(X)} | \chi^{Z \setminus X \cup D(X)})) \\
&= \log(P(\chi^{Z \setminus \{X \cup D(X)\}})P(\chi^{X \cup D(X)} | \chi^{Sib(X) \cup A(X)})) \\
&= \log(P(\chi^{Z \setminus \{X \cup D(X)\}})P(\chi^{X \cup D(X)}, \chi^{Sib(X) \cup A(X)})/P(\chi^{Sib(X) \cup A(X)})) \\
&= \log(P(\chi^{Z \setminus \{X \cup D(X)\}})) + \log(P(\chi^{X \cup Sib(X) \cup D(X) \cup A(X)})) - \log(P(\chi^{Sib(X) \cup P(X)})) \\
&= Q_{Z \setminus \{X \cup D(X)\}} + Q_{X \cup D(X) \cup Sib(X) \cup A(X)} - Q_{Sib(X) \cup A(X)}.
\end{aligned} \tag{25}$$

We can now use Lemma 9 to get

$$\forall z_i \in Z, \quad \beta_{z_i} \log(\mathbb{P}(\chi)) = \log(\mathbb{P}(\chi)).$$

Therefore,

$$\begin{aligned}
\beta \log(\mathbb{P}(\chi)) &= \left(\prod_{z_i \in Z} \beta_{z_i}\right) \log(\mathbb{P}(\chi)) = \left(\prod_{z_i \in Z \setminus \{z_1\}} \beta_{z_i}\right) (\beta_{z_1} \log(\mathbb{P}(\chi))) \\
&= \left(\prod_{z_i \in Z \setminus \{z_1\}} \beta_{z_i}\right) \log(\mathbb{P}(\chi)) = \log(\mathbb{P}(\chi)).
\end{aligned}$$

Thus, $\beta \log(\mathbb{P}(\chi)) = \log(\mathbb{P}(\chi))$.

By Step 6.2, there exists a set of projector $E_{Z \setminus X}$ such that $E_{Z \setminus X} = B_{Z \setminus X} E_{Z \setminus X}$ and

$$\beta = \sum_{X \in F(\chi) \cup \{\emptyset\}} E_{Z \setminus X} = \sum_{X \in F(\chi) \cup \{\emptyset\}} B_{Z \setminus X} E_{Z \setminus X},$$

so

$$\begin{aligned}
\log(\mathbb{P}(\chi)) = \beta \log(\mathbb{P}(\chi)) &= \sum_{X \in F(\chi) \cup \{\emptyset\}} B_{Z \setminus X} E_{Z \setminus X} \log(\mathbb{P}(\chi)) \\
&= \sum_{X \in F(\chi) \cup \{\emptyset\}} E_{Z \setminus X} \log(\mathbb{P}(\chi_{Z \setminus X})).
\end{aligned}$$

Finally, as $\mathbb{P}(\omega = \chi_Z) \neq 0$, with the notation $S(\chi_{Z\setminus X}) = E_{Z\setminus X} \log(\mathbb{P}(\chi_{Z\setminus X}))$, we have the final result:

$$\mathbb{P}(\chi) = \mathbb{P}(\chi_Z) \exp\left(\sum_{X\in F(\chi)} S(\chi_{Z\setminus X})\right).$$

Proof 13 (Proof of Lemma 9) *Let Y be another subset of Z. Applying Equation 20 to Equation 25 yields*

$$B_Y Q_Z = B_Y Q_{Z\setminus\{X\cup D(X)\}} + B_Y Q_{X\cup D(X)\cup Sib(X)\cup A(X)} - B_Y Q_{Sib(X)\cup A(X)},$$

which is equivalent to

$$\frac{\mathbb{P}(\chi_Y)}{\mathbb{P}(\chi_Y^{X\cup Sib(X)\cup D(X)\cup A(X))})} = \frac{\mathbb{P}(\chi_Y^{Z\setminus(X\cup D(X))})}{\mathbb{P}(\chi_Y^{Sib(X)\cup A(X)})}.$$

By applying the same proof as in Lemma 6 for the proof of Equation 22, where we replace ∂X by $Sib(X)\cup A(X)$ and X by $X\cup D(X)$, we have

$$\begin{aligned}&B_{X\cup D(X)} Q_{X\cup Sib(X)\cup D(X)\cup A(X)} - Q_{X\cup Sib(X)\cup D(X)\cup A(X)}\\ &= B_{Z\setminus(X\cup Sib(X)\cup D(X)\cup A(X))} Q_Z - B_{Z\setminus(Sib(X)\cup A(X))} Q_Z.\end{aligned}$$

This equation yields

$$Q_{X\cup Sib(X)} - B_X Q_{X\cup Sib(X)} = B_{Z\setminus Sib(X)} Q_Z - B_{Z\setminus(X\cup Sib(X))} Q_Z. \tag{26}$$

By applying now B_X to Equation 25, we get

$$\begin{aligned}B_{X\cup D(X)} Q_Z &= B_{X\cup D(X)} Q_{Z\setminus(X\cup D(X))} + B_{X\cup D(X)} Q_{X\cup Sib(X)} - B_{X\cup D(X)} Q_{Sib(X)}\\ &= B_{X\cup D(X)} \log(\mathbb{P}(\chi^{Z\setminus(X\cup D(X))})) + B_{X\cup D(X)} Q_{X\cup Sib(X)} - B_{X\cup D(X)} \log(\mathbb{P}(\chi^{Sib(X)}))\\ &= \log(\mathbb{P}(\chi_{X\cup D(X)}^{Z\setminus(X\cup D(X))})) + B_{X\cup D(X)} Q_{X\cup Sib(X)} - \log(\mathbb{P}(\chi_X^{Sib(X)}))\\ &= \log(\mathbb{P}(\chi^{Z\setminus(X\cup D(X))})) + B_{X\cup D(X)} Q_{X\cup Sib(X)} - \log(\mathbb{P}(\chi^{Sib(X)}))\\ &= Q_{Z\setminus(X\cup D(X))} + B_{X\cup D(X)} Q_{X\cup Sib(X)} - Q_{Sib(X)}.\end{aligned} \tag{27}$$

Now, substracting Equation 25 to Equation 27, and using Equation 20,

$$\begin{aligned}Q_Z(\mathbb{1} - B_X) &= B_{X\cup D(X)} Q_{X\cup Sib(X)\cup D(X)\cup A(X)} - Q_{X\cup Sib(X)\cup D(X)\cup A(X)}\\ &= B_{Z\setminus(X\cup Sib(X)\cup D(X)\cup A(X))} Q_Z - B_{Z\setminus(Sib(X)\cup A(X))} Q_Z.\end{aligned}$$

This leads to

$$Q_Z = B_{X\cup Sib(X)} Q_Z + B_{Z\setminus(X\cup Sib(X)\cup A(X)\cup D(X))} Q_Z - B_{Z\setminus(Sib(X)\cup A(X))} Q_Z = \beta_X Q_Z,$$

which ends the proof of Lemma 9.

A.11 Proof of Lemma 7

Proof 14 (Proof of Lemma 7) *According to Corollary 1 and the intermediate result from Proof A.7, an element of the sum is zero if one of the five following conditions is satisfied.*

1. $\bigcup_{k\in K} z_k \neq Z$ *and* $\bigcup_{j\in(J\setminus K)} Z\setminus(z_i\cup\partial z_i) \neq Z$. *i.e.* $\bigcap_{j\in(J\setminus K)}(z_i\cup\partial z_i) \neq \emptyset$
2. $\forall j\in(J\setminus K),\quad X_j \not\subset \bigcup_{j'\in(J\setminus K)} Y_{j'}$.
3. $\forall k_1\in K, \forall k_3\in J\setminus K,\quad \{z_{k_3}\}\not\subset\{z_{k_1}\}$ *i.e.* $z_{k_3}\neq z_{k_1}$,
4. $\forall k_1\in K, \forall k_2\in J\setminus K,\quad z_{k_1}\cup Z\setminus(z_{k_2}\cup\partial z_{k_2}) \neq Z$, *i.e.* $(z_{k_2}\cup\partial z_{k_2}) \neq z_{k_1}$,
5. $\forall (k_2,k_3)\in J\setminus K,\quad z_{k_3}\not\subset Z\setminus(z_{k_2}\cup\partial z_{k_2})$ *i.e.* $z_{k_3}\subset z_{k_2}\cup\partial z_{k_2}$.

1. $\forall (k,j)\in(Z\setminus I)^2$,

$$\begin{aligned}&z_j\cup D(z_j)\not\subset Z\setminus\{z_k\cup Sib(z_k)\cup A(z_k)\cup D(z_k)\}\\ \textit{i.e.}\quad &\{z_j\cup D(z_j)\}\cap\{z_k\cup Sib(z_k)\cup A(z_k)\cup D(z_k)\}\neq\emptyset.\end{aligned}$$

2. $\forall j \in Z \setminus I, \forall i \in I$,

$$z_j \cup D(z_j) \not\subset z_i \cup D(z_i).$$

3. $\forall k \in Z \setminus I$, $\forall j \in I$,

$$\begin{aligned} \{z_j \cup D(z_j)\} \cup Z \backslash \{z_k \cup Sib(z_k) \cup A(z_k) \cup D(z_k)\} &\neq Z \\ \textit{i.e.} \quad \{z_k \cup Sib(z_k) \cup A(z_k) \cup D(z_k)\} &\not\subset \{z_j \cup D(z_j)\}. \end{aligned}$$

4. $\forall (k, k') \in (Z \setminus I)^2$,

$$\begin{aligned} Z \backslash \{z_k \cup Sib(z_k) \cup A(z_k) \cup D(z_j)\} \cup Z \backslash \{z'_k \cup Sib(z'_k) \cup A(z'_k) \cup D(z'_k)\} &\neq Z \\ \textit{i.e.} \quad (z_k \cup Sib(z_k) \cup A(z_k) \cup D(z_k)) \cap (z'_k \cup Sib(z'_k) \cup A(z'_k) \cup D(z'_k)) &\neq \emptyset. \end{aligned}$$

5. $\forall (i, i') \in I^2$,

$$z_i \cup D(z_i) \cup z'_i \cup D(z'_i) \neq Z.$$

6. $\forall (j, j') \in (Z \setminus I)^2$,

$$(\mathbb{1} - B_{z_i \cup D(z_i)})(\mathbb{1} - B_{z'_j \cup D(z'_j)}) \neq 0.$$

Condition 1 implies that every node in $Z \backslash I$ has a relation among each other node's neighbour or descendant or ancestor.

Condition 2 implies that the elements of $Z \backslash I$ are not the descendant of the elements of I.

Condition 3 implies the element of I are not the ancestor of the element of $Z \backslash I$.

Condition 4 implies that every element of $Z \backslash I$ is parent, descendant or neighbour of one another of every other element of $Z \backslash I$.

Condition 5 is satisfied when $Z \backslash I \neq \emptyset$ and at least one element of $Z \backslash I$ is not a descendant of any element of I.

These conditions imply the following fact. Assume that the set $Z \backslash I$ is not totally black. Then, necessarily, $Z \backslash I$ is composed of a set of sibling leaves Sib in addition to their ascendants $D(Sib)$. In other words, it is a filter.

References

[1] Marinka Zitnik, Monica Agrawal, and Jure Leskovec. Modeling polypharmacy side effects with graph convolutional networks. *Bioinformatics*, 34(13):i457–i466, July 2018.

[2] Nanyun Peng, Hoifung Poon, Chris Quirk, Kristina Toutanova, and Wen-tau Yih. Cross-Sentence N-ary Relation Extraction with Graph LSTMs. *Transactions of the Association for Computational Linguistics*, 5:101–115, April 2017.

[3] Muhan Zhang and Yixin Chen. Link Prediction Based on Graph Neural Networks, November 2018. arXiv:1802.09691 [cs, stat].

[4] Michael M. Bronstein, Joan Bruna, Yann LeCun, Arthur Szlam, and Pierre Vandergheynst. Geometric deep learning: going beyond Euclidean data. *IEEE Signal Processing Magazine*, 34(4):18–42, July 2017. arXiv: 1611.08097.

[5] J.M Hammersley and P Clifford. Markov Fields On Finite Graphs And Lattices. 1971.

[6] Keyulu Xu, Weihua Hu, Jure Leskovec, and Stefanie Jegelka. How Powerful are Graph Neural Networks?, February 2019. arXiv:1810.00826 [cs, stat].

[7] Jake Topping, Francesco Di Giovanni, Benjamin Paul Chamberlain, Xiaowen Dong, and Michael M. Bronstein. Understanding over-squashing and bottlenecks on graphs via curvature, March 2022. arXiv:2111.14522 [cs, stat].

[8] M. H. Stone. The Theory of Representation for Boolean Algebras. *Transactions of the American Mathematical Society*, 40(1):37, July 1936.

[9] William G. Bade. On Boolean algebras of projections and algebras of operators. *Transactions of the American Mathematical Society*, 80(2):345–345, February 1955.

[10] Bjarni Jonnson and Alfred Tarski. Boolean Algebras with Operators. *American Journal of Mathematics*, 74(1):127, January 1952.

[11] Werner Ricker. *Operator algebras generated by commuting projections: a vector measure approach*. Number 1711 in Lecture notes in mathematics. Springer, Berlin ; New York, 1999.

[12] H. A. Priestley. Ordered Topological Spaces and the Representation of Distributive Lattices. *Proceedings of the London Mathematical Society*, s3-24(3):507–530, April 1972.

[13] J. H. Rubinstein. Minimal prime ideals and compactifications. *Journal of the Australian Mathematical Society*, 13(4):423–432, June 1972.

[14] M. Hochster. Prime ideal structure in commutative rings. *Transactions of the American Mathematical Society*, 142(0):43–60, 1969.

[15] John D. Lafferty, Andrew McCallum, and Fernando C. N. Pereira. Conditional Random Fields: Probabilistic Models for Segmenting and Labeling Sequence Data. In *Proceedings of the Eighteenth International Conference on Machine Learning*, ICML '01, pages 282–289, San Francisco, CA, USA, June 2001. Morgan Kaufmann Publishers Inc.

[16] Zhiheng Huang, Wei Xu, and Kai Yu. Bidirectional LSTM-CRF Models for Sequence Tagging, August 2015. arXiv:1508.01991 [cs].

[17] Liang-Chieh Chen, George Papandreou, Iasonas Kokkinos, Kevin Murphy, and Alan L. Yuille. DeepLab: Semantic Image Segmentation with Deep Convolutional Nets, Atrous Convolution, and Fully Connected CRFs, May 2017. arXiv:1606.00915 [cs].

[18] Gregory F. Cooper. The computational complexity of probabilistic inference using bayesian belief networks. *Artificial Intelligence*, 42(2):393–405, March 1990.

[19] Dan Roth. On the hardness of approximate reasoning. *Artificial Intelligence*, 82(1-2):273–302, April 1996.

[20] Imre Csiszár and Zsolt Talata. Consistent estimation of the basic neighborhood of Markov random fields. *The Annals of Statistics*, 34(1):123–145, February 2006. Publisher: Institute of Mathematical Statistics.

[21] Shucheng Li, Lingfei Wu, Shiwei Feng, Fangli Xu, Fengyuan Xu, and Sheng Zhong. Graph-to-Tree Neural Networks for Learning Structured Input-Output Translation with Applications to Semantic Parsing and Math Word Problem, October 2020. arXiv:2004.13781 [cs, stat].

[22] Kai Sheng Tai, Richard Socher, and Christopher D. Manning. Improved Semantic Representations From Tree-Structured Long Short-Term Memory Networks, May 2015. arXiv:1503.00075 [cs].

[23] Rajat Talak, Siyi Hu, Lisa Peng, and Luca Carlone. Neural Trees for Learning on Graphs, October 2021. arXiv:2105.07264 [cs].

[24] Edward V. Huntington. Sets of Independent Postulates for the Algebra of Logic. *Transactions of the American Mathematical Society*, 5(3):288–309, 1904. Publisher: American Mathematical Society.

[25] Dietrich Trautmann, Johannes Daxenberger, Christian Stab, Hinrich Schütze, and Iryna Gurevych. Fine-Grained Argument Unit Recognition and Classification. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34(05):9048–9056, April 2020. Number: 05.

[26] Petar Veličković, Guillem Cucurull, Arantxa Casanova, Adriana Romero, Pietro Liò, and Yoshua Bengio. Graph Attention Networks. February 2018.

[27] Jacob Devlin, Ming-Wei Chang, Kenton Lee, and Kristina Toutanova. Bert: Pre-training of deep bidirectional transformers for language understanding. *arXiv preprint arXiv:1810.04805*, 2018.

[28] Thomas Wolf, Lysandre Debut, Victor Sanh, Julien Chaumond, Clement Delangue, Anthony Moi, Pierric Cistac, Tim Rault, Rémi Louf, Morgan Funtowicz, Joe Davison, Sam Shleifer, Patrick von Platen, Clara Ma, Yacine Jernite, Julien Plu, Canwen Xu, Teven Le Scao, Sylvain Gugger, Mariama Drame, Quentin Lhoest, and Alexander M. Rush. Transformers: State-of-the-art natural language processing. In *Proceedings of the 2020 Conference on Empirical Methods in Natural Language Processing: System Demonstrations*, pages 38–45, Online, October 2020. Association for Computational Linguistics.

[29] Nikita Kitaev, Steven Cao, and Dan Klein. Multilingual constituency parsing with self-attention and pre-training. In *Proceedings of the 57th Annual Meeting of the Association for Computational Linguistics*, pages 3499–3505, Florence, Italy, July 2019. Association for Computational Linguistics.

[30] Matthias Fey and Jan E. Lenssen. Fast graph representation learning with PyTorch Geometric. In *ICLR Workshop on Representation Learning on Graphs and Manifolds*, 2019.

[31] Takuya Akiba, Shotaro Sano, Toshihiko Yanase, Takeru Ohta, and Masanori Koyama. Optuna: A next-generation hyperparameter optimization framework. In *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, 2019.

[32] Lisha Li, Kevin Jamieson, Afshin Rostamizadeh, Katya Gonina, Moritz Hardt, Benjamin Recht, and Ameet Talwalkar. Massively parallel hyperparameter tuning, 2018.

[33] Richard M. Karp. Reducibility among combinatorial problems. In Raymond E. Miller, James W. Thatcher, and Jean D. Bohlinger, editors, *Complexity of Computer Computations: Proceedings of a symposium on the Complexity of Computer Computations, held March 20–22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, and sponsored by the Office of Naval Research, Mathematics Program, IBM World Trade Corporation, and the IBM Research Mathematical Sciences Department*, The IBM Research Symposia Series, pages 85–103. Springer US.

[34] M. H. Stone. Subsumption of the Theory of Boolean Algebras under the Theory of Rings. *Proceedings of the National Academy of Sciences*, 21(2):103–105, February 1935. Publisher: Proceedings of the National Academy of Sciences.