EVAPORATION LAW IN KINETIC GRAVITATIONAL SYSTEMS DESCRIBED BY SIMPLIFIED LANDAU MODELS

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Abstract. This paper is devoted to a mathematical and numerical study of a simplified kinetic model for evaporation phenomena in gravitational systems. This is a first step towards a mathematical understanding of more realistic kinetic models in this area. It is well known in the astrophysics literature that the appropriate kinetic model to describe escape (evaporation) from stars clusters is the so-called Vlasov-Landau-Poisson system with vanishing boundary condition at positive microscopic energies. Since collisions between stars and their self-consistent interactions are both taken into account in this model, its mathematical analysis is difficult, and so far not achieved. Here, as a first step, we focus on a simplified framework of this model and make the following assumptions: i) Only homogenous (space-independent) distributions functions are considered, leading to a collisional kinetic model with a vanishing boundary condition in velocity. ii) The interaction potential involved in the Landau collision operator is of Maxwellian type. iii) The escape velocity (or energy) is supposed to be constant. Using these assumptions, we first establish the well-posedness of the associated Cauchy problem. Then, we focus on the long time behavior of the solution and prove that the energy of the system decreases in time as $O(1/\log(t))$ (logarithmic evaporation), with convergence to a Dirac distribution in velocity when time goes to infinity. Finally, a suitable numerical scheme is constructed for this model and some simulations are performed to illustrate the theoretical study.

1. Introduction

1.1. **General model.** At the kinetic level, the evolution of the distribution function $f(x,v,t)$ of a stellar system accounting for evaporation is usually described by the Vlasov-Landau-Poisson equation, see for instance [1, 14, 9, 5]. This model has the following dimensionless form:

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\cdot\nabla_x f - \nabla\phi\cdot\nabla_v f = \\ \qquad \nabla_v\cdot\displaystyle\int |v-v_*|^{\gamma+2}\Pi(v-v_*)\left(f(x,v_*)\nabla_v f(x,v) - f(x,v)\nabla_v f(x,v_*)\right)dv_*, \\ \qquad\qquad \Delta\phi = \displaystyle\int f\,dv, \qquad\qquad \phi(x)\xrightarrow[|x|\to\infty]{} 0, \\ \qquad f(x,v,t=0) = f_0(x,v)\geq 0, \qquad (x,v)\in\mathbb{R}^3\times\mathbb{R}^3, \\ \qquad f(x,v,t) = 0, \qquad\qquad \text{if } e := \dfrac{|v|^2}{2}+\phi(x,t)\geq 0, \end{cases} \tag{1.1}$$

where $\phi(x,t)$ is the gravitational potential and $\Pi(u)$ is the following 3×3 matrix

$$\Pi(u)_{ij} = \frac{|u|^2\delta_{ij}-u_iu_j}{|u|^2}, \tag{1.2}$$

which is also the matrix of the orthogonal projection on $(\mathbb{R}u)^\perp$. The parameter γ may take different values in $[-3, 1]$ depending on the particle interaction law. For power-law interactions in $\frac{1}{r^s}$ we have:

$$\gamma := \frac{s-5}{s-1}.$$

- For Coulombian (plasmas) and Newtonian (stellar systems) interactions, the most interesting but also the most difficult cases, $s = 2$ and $\gamma = -3$.
- For Maxwellian potential, the case we study in this paper, $s = 5$ and $\gamma = 0$.
- Potentials for $0 < \gamma < 1$ ($s > 5$) are called hard potentials.
- Potentials for $-3 < \gamma < 0$ ($2 < s < 5$) are called soft potentials.

This model accounts for the evaporation of stars since the distribution function must vanish wherever $|v|^2/2 + \phi(x,t) > 0$. This is different from the usual Landau equations of plasma physics where no such condition is required since the system is spatially homogeneous (due to electrical neutrality) and the distribution function evolves on the whole velocity domain. The last condition in (1.1) expresses the fact that stars with positive energy are lost by the system. Indeed, the escape velocity of a star at a position x is reached when its microscopic kinetic energy exactly balances its potential energy: $|v_{escape}| = \sqrt{-2\phi(x)}$. The Vlasov-Landau-Poisson system (1.1) is the standard model of stellar dynamics, but to our knowledge, there is no mathematical study (well-posedness, long time behavior, etc) of this model in the literature. The complexity of this model relies on two facts: the coupling with the Poisson equation, and the vanishing boundary condition due to evaporation.

Because of its mathematical complexity, we shall make several simplifying assumptions. We first assume a spatially homogeneous configuration which allows to remove the transport part and the coupling with the Poisson equation. However, we shall take into account the evaporation process in the following manner. In a real cluster, the average value of the squared escape velocity is $\langle v^2_{escape}\rangle = -2\langle\phi(x)\rangle = -2\int \rho\phi\, dx / \int \rho\, dx = -4W/M$ where $W = \frac{1}{2}\int \rho\phi\, dx$ is the potential energy and $M = \int \rho dx$ the mass (the spatial density is $\rho(x) = \int f(x,v)\, dv$). On the other hand, for a power-law interaction in $\frac{1}{r^s}$, the virial theorem reads $2K + (s-1)W = 0$ (for $s \neq 1$) where $K = \frac{1}{2}\int f v^2\, dxdv$ is the kinetic energy (see [1]). Therefore, $\langle v^2_{escape}\rangle = \frac{4}{s-1}\langle v^2\rangle$. For a Maxwellian potential ($s = 5$), this relation becomes $\langle v^2_{escape}\rangle = \langle v^2\rangle$. Therefore, within the homogeneity assumption, we have to solve the Landau equation with the boundary condition $f(t,v) = 0$ for $v > v_{escape}(t)$ where $v_{escape}(t) = \frac{2}{\sqrt{s-1}}\langle v^2\rangle^{1/2}$ and $\langle v^2\rangle = \int f v^2\, dv / \int f dv$. In this paper, we shall replace the time dependent boundary condition by the time independent boundary condition $f(t,v) = 0$ for $v > v_{escape}$ where v_{escape} is a fixed constant. In conclusion, we are led to the homogeneous Landau equation on a bounded velocity domain B_R, which is the open ball of $\mathbb{R}^3$ of radius $R > 0$ centered at the origin. The distribution function f satisfies:

$$\partial_t f(t,v) = Q_R(f,t,v), \tag{1.3}$$

where Q_R is the collision operator given by:

$$Q_R(f,t,v) := \nabla \cdot \int_{B_R} |v - v_*|^{\gamma+2}\Pi(v - v_*)(f(t,v_*)\nabla f(t,v) - f(t,v)\nabla f(t,v_*))\mathrm{d}v_*. \tag{1.4}$$

The divergence and the gradient are taken in the v variable. We add to this equation an initial condition:

$$f(0,.) = f_0 \text{ on } B_R, \tag{1.5}$$

and a Dirichlet condition on the velocity domain:

$$f(t,v) = 0 \text{ for } v \in \partial B_R. \tag{1.6}$$

If $R = +\infty$, we shall refer to this model as the classical Landau equation, which has been widely studied by the mathematical community, see [7] and the references therein. In these works the well-posedness of the Cauchy problem for $R = +\infty$ usually relies on conservation and entropy properties satisfied by the model in this case, namely:

$$\int_{\mathbb{R}^3} Q_\infty(f)(1, v, |v|^2)^T dv = 0, \ \frac{d}{dt}\left(\int_{\mathbb{R}^3} f \log(f) dv\right) = \int_{\mathbb{R}^3} Q_\infty(f)(1+\log(f)) dv \leq 0.$$

In the case $R < +\infty$, which is our concern in this paper, the crucial entropy inequality (second inequality above) is no longer available. Therefore, one has to find new *a priori* estimates to establish the wellposedness of the problem in this case.

In this paper, we study the problem (1.3) in the case of Maxwellian molecules ($\gamma = 0$) and isotropic distribution functions. Note that if we further assume $R = +\infty$, then the problem (1.3)-(1.4) can be solved explicitly in this case, see [12, 7]. Here we focus on the case $R < +\infty$ where no such explicit solutions are known. Our goal is two folds: prove the well-posedness of the problem and establish the $1/\log(t)$ evaporation law for this system.

1.2. **Evaporation laws.** A well known simplifying approximation of the Landau equation (1.3) was proposed in the 1940' by Chandrasekhar [4]. It essentially assumes that the star under consideration has encounters with a separate group of stars having a fixed (usually assumed Maxwellian) velocity distribution, leading to a Fokker-Planck type equation. In other words, the Fokker-Planck equation considered by Chandrasekhar describes the evolution of a "test star" in a bath of "field stars" at statistical equilibrium with fixed temperature (canonical description). By contrast, the Landau equation describes the evolution of the system as a whole and conserves the energy when $R = \infty$ (microcanonical description). Therefore, Chandrasekhar models the evolution of the system by a Fokker-Planck equation of the form:

$$\begin{cases} \dfrac{\partial f}{\partial t}(v,t) = Q_R^{FP}(f)(v) = \nabla \cdot \left[D(|v|)\left(\nabla f(v) + \beta f(v) v\right)\right], & v \in B_R, \\ f(v, t=0) = f_0(v) \geq 0, & v \in B_R, \\ f(v,t) = 0, & \text{if } |v| = R, \end{cases} \tag{1.7}$$

where $D(|v|)$ is some given nonnegative diffusion matrix. The Fokker-Planck equation (1.7) can be derived from the Landau equation (1.3) by making the so-called "thermal bath approximation", i.e. by replacing the function $f(v_*)$ in (1.4) by the Maxwell distribution $f(v) = \rho(2\pi T)^{-3/2}\exp(-v^2/2T)$. Chandrasekhar solved the problem perturbatively and showed that the solution $f_R(v,t)$ goes to zero exponentially as $t \to +\infty$ with a fundamental eigenmode of the form $f_R(v)e^{-|\lambda(R)|t}$. However, there is no quantification of the escape rate $\lambda(R)$ for given $R > 0$ and only its asymptotic value in the limit of large R was exhibited. Recently, this work was complemented by [6], where an exact integral formula for the eigenvalue $\lambda(R)$ of (evaporation rate) was found, for any given R.

Let us come back to the original Landau model (1.3)-(1.4) in which, for all the sequel, we shall take $\gamma = 0$ ($s = 5$). Moreover, we will consider only spherically symmetric distribution functions. This framework is compatible with the invariance of the Landau equation (1.3) under any orthogonal transformation in velocity. With these assumptions, one can derive after elementary calculations a very simple formulation of the Landau model which keeps the whole structure of the original equation. Indeed, denoting

$$g\left(t, \frac{|v|^2}{2}\right) = f(t,v),$$

the collision operator (1.4) becomes

$$\nabla \cdot \int_{B_R} |v - v_*|^2 \Pi(v - v_*)(v g_* \partial_v g - v_* g \partial_{v_*} g_*) \mathrm{d}v_*,$$

where g stands for $g\left(t, \frac{|v|^2}{2}\right)$ and g_* for $g\left(t, \frac{|v_*|}{2}\right)$. Moreover,

$$\begin{aligned}\int_{B_R} |v - v_*|^2 \Pi(v - v_*) v g_* \mathrm{d}v_* &= \int_{B_R} |v - v_*|^2 \Pi(v - v_*) v_* g_* \mathrm{d}v_* \\ &= \int_{B_R} (|v - v_*|^2 v_* - ((v - v_*) \cdot v_*)(v - v_*)) g_* \mathrm{d}v_* \\ &= \int_{B_R} (|v_*|^2 v - (v \cdot v_*) v_*) g_* \mathrm{d}v_* \\ &= \frac{2}{3} \left(\int_{B_R} |v_*|^2 g_* \mathrm{d}v_* \right) v,\end{aligned}$$

$$\begin{aligned}\int_{B_R} |v - v_*|^2 \Pi(v - v_*) v_* \partial_{v_*} g_* \mathrm{d}v_* &= \frac{2}{3} \left(\int_{B_R} |v_*|^2 \partial_{v_*} g_* \mathrm{d}v_* \right) v \\ &= -2 \left(\int_{B_R} g_* \mathrm{d}v_* \right) v.\end{aligned}$$

Rewriting the result with the function f yields, up to the rescaling in time $t' = \frac{2}{3}t$:

$$\partial_t f(t,v) = Q_R(f,f)(t,v) \tag{1.8}$$

with:

$$Q_R(f,g)(t,v) := \nabla \cdot (E_g(t) \nabla f(t,v) + 3v M_g(t) f(t,v)),$$

where M_f and E_f, the mass and the energy of f, are defined by:

$$M_f(t) = \int_{B_R} f(t,v) \mathrm{d}v, \qquad E_f(t) = \int_{B_R} f(t,v) |v|^2 \mathrm{d}v. \tag{1.9}$$

We shall see that this model gives rise to completely different behaviors compared to the Chandrasekhar model described above.

Remark 1.1. This type of collision operator has already been introduced for instance in [3] in the case of a Vlasov-Fokker-Planck equation, the difference here is the homogeneous Dirichlet condition in the velocity variable.

Remark 1.2. As discussed in [10, 6], when spatial inhomogeneity is ignored, the Chandrasekhar model (1.7) provides a better physical description of star escape than the homogeneous Landau model (1.3). This is because the thermal bath approximation takes into account the equilibration between evaporation (leading to cooling) and core concentration (leading to heating) resulting in an isothermal distribution. Therefore, the real mass decay of star clusters is exponential as in the Chandrasekhar model. However, on a mathematical point of view, the study of the homogeneous Landau model provides a first step towards the general inhomogeneous model (1.1) and is therefore interesting in this respect.

In this paper we first prove the existence and uniqueness of classical solution to the Cauchy problem (1.8)-(1.5)-(1.6), if the initial condition is sufficiently regular. Then we prove that the mass of the solution decreases to a positive constant and that its energy decreases to 0, and give their exact asymptotic behavior for large time. In particular we shall prove that the evaporation shape of the system (when modeled by (1.8)-(1.5)-(1.6)) is in $O(1/\log(t))$ for large time, in contrast with the Chandrasekhar model (1.7) where the evaporation holds exponentially. This leads to the convergence of $f(v,t)$ to a Dirac distribution in velocity when time goes to infinity. In the last part of this paper, we construct a suitable numerical discretization that inherits the main properties from the continuous model and whose simulations clearly illustrate our theoretical results. This is a first step towards the mathematical study of the more physically relevent model (1.1).

1.3. **Notations and definitions.** We list here some notations and definitions that will be used throughout this paper.

Our classical solutions are defined on the closed domain:

$$D := [0,+\infty[\times\overline{B_R}.$$

We shall also work on a bounded-in-time domain, to establish *a priori* estimates on a finite time interval $[0,T]$. The domain of definition becomes:

$$D_T := [0,T]\times\overline{B_R}.$$

Classical solutions are functions of $\mathcal{C}^{1,2}(D)$ or $\mathcal{C}^{1,2}(D_T)$ i.e. functions on D or D_T which are $\mathcal{C}^1$ in the time variable and $\mathcal{C}^2$ in the velocity variable and that verify Eq. (1.8) in the classical sense. However, to prove the existence of classical solutions we need more regularity and we work with Friedman's Hölder-type Banach space $B^\ell(D_T)$ for $\ell\in\mathbb{R}_+^*\setminus\mathbb{N}$, defined by the norm (see [8]):

$$\begin{aligned}\|f\|_{B^\ell} := & \sup_{\substack{0<t<T\\ v\in B_R}}\sum_{|\alpha|+2r\le[\ell]}|\partial_t^r\partial_v^\alpha f(t,v)|\\ &+\sup_{\substack{0<t<T\\ v,w\in B_R, v\neq w}}\sum_{|\alpha|+2r=[\ell]}\frac{|\partial_t^r\partial_v^\alpha f(t,v)-\partial_t^r\partial_v^\alpha f(t,w)|}{|v-w|^{\ell-[\ell]}}\\ &+\sup_{\substack{0<s<t<T\\ v\in B_R}}\sum_{|\alpha|+2r=[\ell]}\frac{|\partial_t^r\partial_v^\alpha f(t,v)-\partial_t^r\partial_v^\alpha f(s,v)|}{|t-s|^{\frac{\ell-[\ell]}{2}}}\end{aligned}$$

where $[\ell]$ is the integer part of ℓ and $\alpha\in\mathbb{N}^3$. The space $B^\ell(\overline{B_R})$ is defined with a similar norm in which the t variable is omitted. We shall also consider weak solutions in the space:

$$X_T := \mathcal{C}([0,T],\mathbb{L}^2(B_R))\cap\mathbb{L}^2([0,T],H_0^1(B_R))\cap H^1(]0,T[,H^{-1}(B_R)), \qquad (1.10)$$

with the norm:

$$|||f|||_T := \|f\|_{\mathbb{L}^\infty([0,T],\mathbb{L}^2(B_R))}+\|\nabla f\|_{\mathbb{L}^2([0,T],\mathbb{L}^2(B_R))}+\|\partial_t f\|_{\mathbb{L}^2([0,T],H^{-1}(B_R))}. \quad (1.11)$$

By weak solutions, we mean solutions satisfying equation (1.8) for a.e. t in $H^{-1}(B_R)$, the condition (1.6) for a.e. t in the sense of traces and (1.5) in $\mathbb{L}^2(B_R)$.

2. **Theoretical Results**

2.1. **Main result.** Our main result in this paper is the following:

Theorem 2.1. *Let $\ell \in \mathbb{R}_+^* \setminus \mathbb{N}$ and let $f_0 \in B^{\ell+2}(B_R)$ be a nonnegative and nonzero initial condition. Then, there exists a unique nonnegative classical solution f in $B^{\ell+2}(D)$ to the problem* (1.8), (1.6), (1.5).

Moreover, the mass and energy of f defined by (1.9) *are $\mathcal{C}^1$, positive, nonincreasing functions, and satisfy:*

(i) *a conservation law:*

$$\forall t \in [0, +\infty[\quad R^2 M_f(t) - E_f(t) = R^2 M_{f_0} - E_{f_0} > 0, \tag{2.1}$$

(ii) *a non-zero mass limit:*

$$\lim_{t\to+\infty} M_f(t) = M_{f_0} - \frac{E_{f_0}}{R^2} =: M_\infty > 0, \tag{2.2}$$

(iii) *a logarithmic evaporation:*

$$E_f(t) \underset{t\to+\infty}{\sim} \frac{3M_\infty R^2}{2\log t}. \tag{2.3}$$

In particular, the distribution function $f(t,.)$ converges narrowly as $t \to +\infty$ to the Dirac distribution function $M_\infty \delta_{v=0}$.

Remark 2.2. We do not ask f_0 to be radial since it is not necessary to prove the result, but of course the result given by Theorem 2.1 in the case where f_0 is not radial has no physical meaning because it does not correspond to the model.

Remark 2.3. In a future work [2], we will study the self-similar behavior of the solution f. It appears that f admits a Maxwellian self-similar profile.

The proof of Theorem 2.1 is given in Section 2.3. In Section 2.2 we give some estimates that are useful in Section 2.3.1 to prove existence and uniqueness of solutions. The proof of (2.3) is divided in two parts: in Section 2.3.2 we prove that:

$$\limsup_{t\to+\infty} E_f(t)\log t \le \frac{3M_\infty R^2}{2}, \tag{2.4}$$

then in Section 2.3.3 we prove that:

$$\liminf_{t\to+\infty} E_f(t)\log t \ge \frac{3M_\infty R^2}{2}. \tag{2.5}$$

The end of the proof is given in Section 2.3.4.

2.2. **Basic estimates.** To prove existence and uniqueness of weak solutions, we need some basic estimates which we develop in this section.

2.2.1. *Ellipticity.* To prove the ellipticity of the parabolic operator, we need a lower bound on the energy. This is obtained in two steps: First, Lemma 2.4 says that a positive lower bound for the energy can be found from a positive lower bound of the mass and an upper bound of the $\mathbb{L}^2$ norm. Second, we use the conservation law (2.1) to get a lower bound for the mass. In Lemma 2.5 we prove this minoration even in the more general case where the mass and the energy involved in the model are computed from a given function g that can be different from the solution f. This slight extension will be used in our fixed point proof.

Lemma 2.4. *Let $f \in \mathbb{L}^2(B_R)$ be a nonnegative and nonzero function. Then there exists a universal constant $C > 0$ such that:*

$$E_f \ge C\|f\|_{\mathbb{L}^2}^{-\frac{4}{3}} M_f^{\frac{7}{3}}.$$

Proof. Setting:

$$r := \left(\frac{4\pi}{3}\right)^{-\frac{1}{7}} \|f\|_{\mathbb{L}^2}^{-\frac{2}{7}} E_f^{\frac{2}{7}}.$$

we have:

$$\begin{aligned} M_f = \int_{B_r} f(v)\mathrm{d}v + \int_{B_R\setminus B_r} f(v)\mathrm{d}v &\leq |B_r|^{\frac{1}{2}}\|f\|_{\mathbb{L}^2} + \frac{1}{r^2}E_f \\ &\leq 2\left(\frac{4\pi}{3}\right)^{\frac{2}{7}} E_f^{\frac{3}{7}}\|f\|_{\mathbb{L}^2}^{\frac{4}{7}} \end{aligned}$$

and the result follows. □

Lemma 2.5. *Let $T > 0$, $g \in \mathcal{C}([0,T],\mathbb{L}^2(B_R))$ be a nonnegative function and let f be a function in $\mathcal{C}^{1,2}(D_T)$ which satisfies* (1.6), (1.5), *where f_0 is a nonzero and nonnegative function in $\mathbb{L}^2(B_R)$. Assume in addition that:*

$$\partial_t f = Q_R(f,g). \tag{2.6}$$

Then, f is nonnegative and we have:

$$R^2M_f(t) - E_f(t) \geq (R^2M_{f_0} - E_{f_0})e^{-6\int_0^T M_g(s)\mathrm{d}s}.$$

In particular, M_f is bounded from below by a positive constant. Moreover, the function $t \mapsto M_f(t)$ is nonincreasing. Finally, if we assume additionnally that $f = g$, then we have the conservation law (2.1) *and the function $t \mapsto E_f(t)$ is also nonincreasing.*

Proof. It is classical (see [8] for instance) that f is nonnegative as a solution to a linear parabolic equation with continous coefficients, homogeneous Dirichlet boundary condition and nonnegative initial condition. We now compute the derivatives of M_f and E_f. We have

$$\begin{aligned} M_f'(t) &= \int_{B_R} \nabla\cdot(E_g(t)\nabla f(t,v) + 3vM_g(t)f(t,v))\mathrm{d}v \\ &= E_g(t)\int_{\partial B_R} \nabla f(t,v)\cdot\frac{v}{|v|}\mathrm{d}\sigma(v), \end{aligned}$$

with σ the measure on ∂B_R. This last quantity is nonpositive because f is nonnegative on B_R and vanishes on ∂B_R and then the scalar product $\nabla f(t,.)\cdot\frac{v}{|v|}$ is nonpositive. Thus, the mass is nonincreasing. Next, we have

$$\begin{aligned} E_f'(t) &= \int_{B_R} \nabla\cdot(E_g(t)\nabla f(t,v) + 3vM_g(t)f(t,v))v^2\mathrm{d}v \\ &= R^2E_g(t)\int_{\partial B_R} \nabla f(t,v)\cdot\frac{v}{|v|}\mathrm{d}\sigma(v) \\ &\quad - 2\int_{B_R}(E_g(t)\nabla f(t,v)\cdot v + 3M_g(t)f(t,v)v^2)\mathrm{d}v. \end{aligned}$$

An integration by parts yields

$$\int_{B_R} \nabla f(t,v)\cdot v\mathrm{d}v = -3\int_{B_R} f(t,v)\mathrm{d}v,$$

and then

$$E_f'(t) = R^2E_g(t)\int_{\partial B_R}\nabla f(t,v)\cdot\frac{v}{|v|}\mathrm{d}\sigma(v) + 6(M_f(t)E_g(t) - M_g(t)E_f(t)).$$

In the special case where $f = g$, we have

$$E'_f(t) = R^2 E_f(t) \int_{\partial B_R} \nabla f(t,v) \cdot \frac{v}{|v|} \mathrm{d}\sigma(v) = R^2 M'_f(t) \tag{2.7}$$

from which we deduce (2.1). In the general case, we only have

$$\begin{aligned} R^2 M'_f(t) - E'_f(t) &= -6(M_f(t)E_g(t) - M_g(t)E_f(t)) \\ &\geq -6M_g(t)(R^2 M_f(t) - E_f(t)), \end{aligned}$$

and the use of Gronwall's lemma completes the proof. □

2.2.2. $\mathbb{L}^2$*-estimates.* We use classical $\mathbb{L}^2$-estimates techniques for parabolic operators to derive the necessary estimates that will be used in fixed point proof. We give, for the solutions of (2.6), $\mathbb{L}^\infty(\mathbb{L}^2)$-estimates and estimates for the norm $|\!|\!|\cdot|\!|\!|_T$ defined by (1.11).

Lemma 2.6. *Let $T > 0$ and f, f_0 and g be as in Lemma 2.5. Then we have the two following results.*

(i) *Let M be an upper bound of M_g, then there exists a positive constant C which depends only on T and M such that*

$$\|f\|_{\mathbb{L}^\infty(\mathbb{L}^2)} \leq C\|f_0\|_{\mathbb{L}^2}.$$

(ii) *Let β be an upper bound of $\|g\|_{\mathbb{L}^\infty(\mathbb{L}^2)}$ and suppose that there exists a positive lower bound m of M_g. Then there exists a positive constant C which depends only on R, T, m and β such that*

$$|\!|\!|f|\!|\!|_T \leq C\|f_0\|_{\mathbb{L}^2}.$$

Remark 2.7. As a corollary of this Lemma together with Lemma 2.5, we infer that any solution f of (1.8), (1.5) and (1.6) satisfies

$$M_f(t) \leq M_{f_0}, \quad \|f\|_{\mathbb{L}^\infty(\mathbb{L}^2)} \leq \|f_0\|_{\mathbb{L}^2}. \tag{2.8}$$

Proof. Let us prove (i). We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R} f^2(t,v)\mathrm{d}v &= 2\int_{B_R} f(t,v)\nabla\cdot(E_g(t)\nabla f(t,v) + 3vM_g(t)f(t,v))\mathrm{d}v \\ &= -2E_g(t)\int_{B_R} |\nabla f(t,v)|^2 \mathrm{d}v - 6M_g(t)\int_{B_R} f(t,v)\nabla f(t,v)\cdot v\mathrm{d}v \\ &= -2E_g(t)\int_{B_R} |\nabla f(t,v)|^2 \mathrm{d}v - 3M_g(t)\int_{B_R} \nabla(f^2(t,v))\cdot v\mathrm{d}v. \end{aligned}$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R} f^2(t,v)\mathrm{d}v = -2E_g(t)\int_{B_R} |\nabla f(t,v)|^2\mathrm{d}v + 9M_g(t)\int_{B_R} f^2(t,v)\mathrm{d}v. \tag{2.9}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|f(t,.)\|^2_{\mathbb{L}^2}\right) \leq 9M\|f(t,.)\|^2_{\mathbb{L}^2}$$

and (i) is proved by Gronwall's lemma. Now let us prove (ii). First we use (i) with $M := \beta|B_R|^{\frac{1}{2}}$. Then, from Lemma 2.4, there exists a positive constant k such that

$$E_g \geq k\beta^{-\frac{4}{3}}m^{\frac{7}{3}}.$$

Integrating (2.9) yields

$$\|\nabla f\|_{\mathbb{L}^2([0,T]\times B_R)} \leq C_1\|f_0\|_{\mathbb{L}^2(B_R)}, \tag{2.10}$$

where C_1 is a constant which depends only on R, T, m and β. Now let us take $\varphi \in H_0^1(B_R)$. As f is solution of (2.6), we have

$$\int_{B_R} \partial_t f(t,v)\varphi(v)\mathrm{d}v = -E_g(t)\int_{B_R} \nabla f(t,v)\cdot\nabla\varphi(v)\mathrm{d}v$$
$$- 3M_g(t)\int_{B_R} f(t,v)v\cdot\nabla\varphi(v)\mathrm{d}v.$$

By Cauchy-Schwarz, $E_g(t) \leq R^2 M_g(t) \leq R^2|B_R|^{\frac{1}{2}}\beta$. Moreover, by the Poincaré's inequality, there exists a constant C_R such that $\|f(t,.)\|_{\mathbb{L}^2} \leq C_R\|\nabla f(t,.)\|_{\mathbb{L}^2}$. Hence we have

$$\|\partial_t f(t,.)\|_{H^{-1}} \leq C_2\|\nabla f(t,.)\|_{\mathbb{L}^2}, \tag{2.11}$$

with C_2 a constant which depends only on R, T, m and β. Finally, Item (ii) is proved by integrating (2.11) between 0 and T and using (i) and (2.10). □

Remark 2.8. The estimates used to prove (2.11) can be written in the following way:

$$\left|\int_{B_R} Q_R(f,g)(t,v)h(t,v)\mathrm{d}v\right| \leq C_R\|\nabla f(t,.)\|_{\mathbb{L}^2}\|g(t,.)\|_{\mathbb{L}^1}\|\nabla h(t,.)\|_{\mathbb{L}^2}, \tag{2.12}$$

with C_R depending only on R.

2.3. **Proof of the main result.** This section is devoted to the proof of Theorem 2.1.

2.3.1. *Existence and uniqueness of a classical solution.* Let $T > 0$ be fixed once for all and let $f_0 \in B^{\ell+2}$, with $\ell \in \mathbb{R}_+^* \setminus \mathbb{N}$, be nonzero and nonnegative. Estimates of Lemmas 2.5 and 2.6 allow us to prove the existence and uniqueness of a classical solution of (1.8), (1.6), (1.5) on $[0,T]$, using Picard-Banach fixed point theorem. We use the norm $|\!|\!|\cdot|\!|\!|_T$ defined by (1.11). To obtain a contraction mapping we shall work on an interval $[0,T_0]$ where $T_0 \in [0,T]$ whose value will be chosen later. We know from the first part of Lemma 2.6 that there exists a constant C_1 which depends only on T and M_{f_0} such that if f and g satisfy the assumptions of Lemma 2.5 and if $M_g \leq M_{f_0}$ then

$$\|f\|_{\mathbb{L}^\infty(\mathbb{L}^2)} \leq C_1\|f_0\|_{\mathbb{L}^2}.$$

Let

$$m := M_\infty e^{-6M_{f_0}T},$$

with M_∞ defined by (2.2) and

$$\beta := \max(C_1\|f_0\|_{\mathbb{L}^2}, M_{f_0}|B_R|^{-\frac{1}{2}}).$$

From the second part of Lemma 2.6, there exists a constant C_2 which depends only on R, T and $\|f_0\|_{\mathbb{L}^2}$ such that, if $\|g\|_{\mathbb{L}^\infty(\mathbb{L}^2)} \leq \beta$ and $M_g \geq m$,

$$|\!|\!|f|\!|\!|_{T_0} \leq C_2\|f_0\|_{\mathbb{L}^2}.$$

Let

$$K := \max(C_2\|f_0\|_{\mathbb{L}^2}, \beta),$$

and consider the set F defined by

$$F := \{g \in B^{\ell+2}(D_{T_0}) : g \text{ nonnegative}, m \leq M_g \leq M_{f_0}, \|g\|_{\mathbb{L}^\infty(\mathbb{L}^2)} \leq \beta, |\!|\!|g|\!|\!|_{T_0} \leq K\}.$$

For T_0 small enough, it is nonempty because, as $\beta \geq M_{f_0}|B_R|^{-\frac{1}{2}}$ and $K \geq \beta$, it contains some constant functions. For $g \in F$, M_g and E_g are Hölder continuous functions of time and we know from Lemma 2.4 that E_g is bounded from below by a positive constant λ which depends only on R, T, $\|f_0\|_{\mathbb{L}^2}$ and $R^2M_{f_0} - E_{f_0}$. Thus, from classical existence theorem for linear parabolic equations (see [11] or [8]), we know that there exists a unique solution $\Phi(g) \in B^{\ell+2}(D_T)$ to

$$\partial_t(\Phi(g))(t,v) = Q_R(\Phi(g),g)(t,v)$$

which satisfies (1.5) and (1.6). From Lemmas 2.5 and 2.6 we have $\Phi(g) \in F$. Let $g, \tilde{g} \in F$, then $\Phi(g) - \Phi(\tilde{g})$ satisfies

$$\begin{gathered}\partial_t(\Phi(g) - \Phi(\tilde{g})) - Q_R(\Phi(g) - \Phi(\tilde{g}), g) = Q_R(\Phi(\tilde{g}), g - \tilde{g}), \\ \forall (t,v) \in [0,T] \times \partial B_R \quad (\Phi(g) - \Phi(\tilde{g}))(t,v) = 0, \\ (\Phi(g) - \Phi(\tilde{g}))(0,.) = 0.\end{gathered} \tag{2.13}$$

For convenience, let us denote $\psi := \Phi(g) - \Phi(\tilde{g})$ and $\chi := g - \tilde{g}$. The calculation used in the proof of Lemma 2.6 leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R} \psi(t,v)^2 \mathrm{d}v = -2E_g(t)\int_{B_R} |\nabla\psi(t,v)|^2 \mathrm{d}v + 9M_g(t)\int_{B_R} \psi^2(t,v)\mathrm{d}v \\ + \int_{B_R} Q_R(\Phi(\tilde{g}),\chi)(t,v)\psi(t,v)\mathrm{d}v.\end{aligned}$$

Moreover by (2.12), there exists a constant C_3 which depends only on R such that

$$\begin{aligned}\left|\int_{B_R} Q_R(\Phi(\tilde{g}),\chi)(t,v)\psi(t,v)\mathrm{d}v\right| &\le C_3 \|\nabla\Phi(\tilde{g})(t,.)\|_{\mathbb{L}^2} \, |\!|\!|\chi|\!|\!|_{T_0} \, \|\nabla\psi(t,.)\|_{\mathbb{L}^2} \\ &\le \lambda\|\nabla\psi(t,.)\|^2_{\mathbb{L}^2} + \frac{C_3^2}{4\lambda} |\!|\!|\chi|\!|\!|^2_{T_0} \, \|\nabla\Phi(\tilde{g})(t,.)\|^2_{\mathbb{L}^2}.\end{aligned}$$

Hence, using $E_g \ge \lambda > 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\psi(t,.)\|^2_{\mathbb{L}^2}\right) + \lambda\|\nabla\psi(t,.)\|^2_{\mathbb{L}^2} \le \frac{C_3^2}{4\lambda} |\!|\!|\chi|\!|\!|^2_{T_0} \, \|\nabla\Phi(\tilde{g})(t,.)\|^2_{\mathbb{L}^2} + 9M_{f_0}\|\psi(t,.)\|^2_{\mathbb{L}^2}. \tag{2.14}$$

By Gronwall's Lemma, as $\psi(0,.) = 0$,

$$\|\psi\|^2_{\mathbb{L}^\infty(\mathbb{L}^2)} \le \frac{C_3^2}{4\lambda}\|\nabla\Phi(\tilde{g})\|^2_{\mathbb{L}^2} e^{9M_{f_0}T} \, |\!|\!|\chi|\!|\!|^2_{T_0}. \tag{2.15}$$

Then, using again (2.14),

$$\|\nabla\psi\|_{\mathbb{L}^2} \le C_4\|\nabla\Phi(\tilde{g})\|_{\mathbb{L}^2} \, |\!|\!|\chi|\!|\!|_{T_0}, \tag{2.16}$$

with C_4 a constant which depends only on $\|f_0\|_{\mathbb{L}^2}$ and $R^2M_{f_0} - E_{f_0}$.

Now let us fix a function $g_0 \in F$. Applying (2.16) with $(\tilde{g}, g_0)$ instead of $(g, \tilde{g})$ leads to

$$\begin{aligned}\|\nabla\Phi(\tilde{g})\|_{\mathbb{L}^2} &\le \|\nabla\Phi(g_0)\|_{\mathbb{L}^2}(1 + C_4 \, |\!|\!|\tilde{g} - g_0|\!|\!|_{T_0}) \\ &\le \|\nabla\Phi(g_0)\|_{\mathbb{L}^2}(1 + 2C_4K).\end{aligned} \tag{2.17}$$

Finally, as ψ is solution to (2.13), using (2.12), (2.15) and (2.17) we deduce that there exists a constant C_5 which depends only on $\|f_0\|_{\mathbb{L}^2}$ and $R^2M_{f_0} - E_{f_0}$ such that

$$|\!|\!|\Phi(g) - \Phi(\tilde{g})|\!|\!|_{T_0} \le C_5\|\nabla\Phi(g_0)\|_{\mathbb{L}^2([0,T_0]\times B_R)} \, |\!|\!|g - \tilde{g}|\!|\!|_{T_0}.$$

Now we can choose T_0 such that

$$C_5\|\nabla\Phi(g_0)\|_{\mathbb{L}^2([0,T_0]\times B_R)} \le \frac{1}{2}.$$

With this choice of T_0, Φ is a contraction mapping in the $|\!|\!|\cdot|\!|\!|_{T_0}$ norm. The set F is not closed $|\!|\!|\cdot|\!|\!|_{T_0}$, we cannot apply directly Picard-Banach theorem. Nevertheless, taking any $g_0 \in F$ and defining, for $n \in \mathbb{N}$,

$$g_{n+1} := \Phi(g_n),$$

we obtain a sequence which converges to $f \in X_{T_0}$. It is clear that f is a weak solution of (1.8). We know from regularity results for linear parabolic equations of [11] that in fact $f \in H^1([0,T_0], \mathbb{L}^2(B_R))$ and using the Sobolev embedding

$$H^1([0,T_0], \mathbb{L}^2(B_R)) \hookrightarrow \mathcal{C}^{0,\frac{1}{2}}([0,T_0], \mathbb{L}^2(B_R)),$$

we see that M_f and E_f are $\frac{1}{2}$-Hölder continuous. Therefore, thanks to the Hölder regularity results for linear parabolic equations (see [8, 11]) we conclude that f is a classical solution in $B^{\min(\ell,1)+2}(D_{T_0})$. Then, by a bootstrap argument, since the Hölder regularity of the coefficients of (1.8) is the same as the Hölder regularity in the t variable of f, $f \in B^{\ell+2}(D_{T_0})$ and f is a classical solution to (1.8). Thus, we have constructed a classical solution on $[0,T_0]$. This solution is unique because, if $\tilde{f}$ is a classical solution then $\tilde{f} \in F$ and $\Phi(\tilde{f}) = \tilde{f}$ and then $\tilde{f} = f$. This proves the existence of a local solution but also the fact that the set of $t \in [0,T]$ such that there exists a unique solution on $[0,t]$ is open and non empty. This set is also closed thanks to the global estimates on $[0,T]$ in the weak norm $\left|\!\left|\!\left| \cdot \right|\!\right|\!\right|_T$, the injection

$$\mathbb{L}^2([0,t], H_0^1(B_R)) \cap H^1(]0,t[, H^{-1}(B_R)) \hookrightarrow \mathcal{C}([0,t], \mathbb{L}^2(B_R)),$$

and the previous argument used to recover the Hölder regularity. Thus we can construct a solution on $[0,T]$ and, as T is arbitrary, we conclude that there exists a unique solution f on D.

If f_0 is radial then f is radial. Indeed, if ρ is a orthogonal transformation of $\mathbb{R}^3$ and if g is defined on D by

$$g(t,v) = f(t,\rho(v)).$$

Then we have

$$\begin{aligned}
\partial_t g(t,v) &= \partial_t f(t,\rho(v)) \\
&= E_f(t)\triangle f(t,\rho(v)) + 3M_f(t)\rho(v)\cdot\nabla f(t,\rho(v)) + 9M_f(t)f(t,\rho(v)) \\
&= E_g(t)\triangle g(t,v) + 3M_g(t)\rho(v)\cdot\rho(\nabla g(t,v)) + 9M_g(t)g(t,v) \\
&= Q_R(g,g)(t,v),
\end{aligned}$$

$$g(0,v) = f_0(\rho(v)) = f_0(v)$$

and for all $v \in \partial B_R$

$$g(t,v) = f(t,\rho(v)) = 0.$$

As f is unique, we have $f = g$ for any ρ and then f is radial.

2.3.2. *Asymptotic upper bound for the energy.* In this section we prove the upper bound (2.4). It is rather intuitive that when time goes to infinity the energy must converge to 0 as time goes to ∞ and that the solution must converge to a Dirac function at 0 with the mass M_∞ defined in Theorem 2.1 . Indeed, the diffusion term with the homogeneous Dirichlet boundary condition contributes to the evaporation of particles with highest velocity and the friction term slow down all particles. Thus it should not remain particles with a nonzero velocity. The decrease of the energy can be controled by the decrease of the $\mathbb{L}^2$ norm, and it is tempting to adapt the proof of the exponential decrease of the $\mathbb{L}^2$ norm of solution to the heat equation. Actually, it is this kind of idea we use here.

In a first step, we prove that for all $T \geq 0$ there exists a positive constant C which depends only on R and T such that, for all $t \geq T$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_T^t \frac{E_f(s)}{4R^2} e^{-\frac{3M_f(T)R^2}{2E_f(s)}}\,\mathrm{d}s\right) \leq C\exp\left(-\int_T^t \frac{E_f(s)}{4R^2} e^{-\frac{3M_f(T)R^2}{2E_f(s)}}\,\mathrm{d}s\right). \tag{2.18}$$

It is clear from Lemma 2.4 and Lemma 2.5 that E_f does not vanish in finite time. Let $T \geq 0$. Consider, on $[T, +\infty[$, the weighted $\mathbb{L}^2$ norm

$$F(t) := \int_{B_R} f^2(t,v) w(t,v) \mathrm{d}v,$$

with

$$w(t,v) := \exp\left(\frac{3}{2E_f(t)}(v^2 M_f(t) - R^2 M_f(T))\right).$$

We compute the derivative of F,

$$\begin{aligned} F'(t) &= 2\int_{B_R} f(t,v)\partial_t f(t,v) w(t,v)\mathrm{d}v \\ &+ \int_{B_R} f^2(t,v)\frac{3}{2}\frac{M_f'(t)v^2 E_f(t) - (v^2 M_f(t) - R^2 M_f(T))E_f'(t)}{E_f^2(t)} w(t,v)\mathrm{d}v. \end{aligned}$$

Using that M_f and E_f are positive and nonincreasing (Lemma 2.5) and that $t \geq T$, we can discard the second term and get

$$F'(t) \leq 2\int_{B_R} f(t,v)\partial_t f(t,v) w(t,v)\mathrm{d}v.$$

Writing $Q_R(f,f)$ in a diffusive form, we have

$$\partial_t f(t,v) = E_f(t)\nabla\cdot\left(e^{-\frac{3M_f(t)v^2}{2E_f(t)}}\nabla\left(e^{\frac{3M_f(t)v^2}{2E_f(t)}} f(t,v)\right)\right),$$

thus

$$F'(t) \leq 2E_f(t)\int_{B_R} f(t,v)\nabla\cdot\left(e^{-\frac{3M_f(t)v^2}{2E_f(t)}}\nabla\left(e^{\frac{3M_f(t)v^2}{2E_f(t)}} f(t,v)\right)\right) w(t,v)\mathrm{d}v.$$

An integration by part leads to

$$\begin{aligned} F'(t) &\leq -2E_f(t) e^{-\frac{3M_f(T)R^2}{2E_f(t)}} \int_{B_R} e^{-\frac{3M_f(t)v^2}{2E_f(t)}} \left|\nabla\left(e^{\frac{3M_f(t)v^2}{2E_f(t)}} f(t,v)\right)\right|^2 \mathrm{d}v \\ &\leq -2E_f(t) e^{-\frac{3M_f(T)R^2}{E_f(t)}} \int_{B_R} \left|\nabla\left(e^{\frac{3M_f(t)v^2}{2E_f(t)}} f(t,v)\right)\right|^2 \mathrm{d}v. \end{aligned}$$

Then we use Poincaré's inequality and get

$$\begin{aligned} F'(t) &\leq -\frac{E_f(t)}{2R^2} e^{-\frac{3M_f(T)R^2}{E_f(t)}} \int_{B_R} e^{\frac{3M_f(t)v^2}{E_f(t)}} f^2(t,v)\mathrm{d}v \\ &\leq -\frac{E_f(t)}{2R^2} e^{-\frac{3M_f(T)R^2}{2E_f(t)}} F(t). \end{aligned}$$

Grownwall's lemma leads to

$$F(t) \leq F(T)\exp\left(\int_T^t -\frac{E_f(s)}{2R^2} e^{-\frac{3M_{f_0}R^2}{2E_f(s)}} \mathrm{d}s\right).$$

Finally, using that

$$\|f(t,.)\|_{\mathbb{L}^2} \leq e^{\frac{3M_f(T)R^2}{4E_f(t)}} \sqrt{F(t)} \leq e^{\frac{3M_f(T)R^2}{2E_f(t)}} \sqrt{F(t)},$$

$$E_f(t) \leq \left(\int_{B_R} |v|^4 \mathrm{d}v\right)^{\frac{1}{2}} \|f(t,.)\|_{\mathbb{L}^2},$$

we get (2.18).

Let us now deduce (2.4) from (2.18). Defining, for $t \geq T$,

$$G(t) := \int_T^t \frac{E_f(s)}{4R^2} e^{-\frac{3M_f(T)R^2}{2E_f(s)}} \mathrm{d}s,$$

we have from (2.18)

$$G'(t)e^{G(t)} \leq C.$$

Thus

$$G(t) \leq \log(C(t-T)+1).$$

As E_f is nonincreasing, the function

$$t \mapsto \frac{E_f(t)}{4R^2} e^{-\frac{3M_f(T)R^2}{2E_f(t)}}$$

is nonincreasing. Thus

$$\begin{aligned}(t-T)\frac{E_f(t)}{4R^2} e^{-\frac{3M_f(T)R^2}{2E_f(t)}} &\leq \log(C(t-T)+1)\\ \frac{E_f(t)}{4R^2} e^{-\frac{3M_f(T)R^2}{2E_f(t)}} &\leq \frac{\log(C(t-T)+1)}{t-T}.\end{aligned}$$

Let $\varepsilon > 0$ and k defined by

$$k := \min_{x\in\mathbb{R}_+} \frac{x}{4R^2} e^{\frac{\varepsilon}{x}} = \frac{\varepsilon e}{4R^2}.$$

Then k is positive, depends only on R and ε and we have for all $t \geq T$

$$k \leq \frac{E_f(t)}{4R^2} e^{\frac{\varepsilon}{E_f(t)}}.$$

Thus, for t sufficiently large, we have

$$\begin{gathered} e^{-\frac{1}{E_f(t)}\left(\frac{3M_f(T)R^2}{2}+\varepsilon\right)} \leq \frac{\log(C(t-T)+1)}{k(t-T)} \\ E_f(t)\log t \leq \left(\frac{3M_f(T)R^2}{2}+\varepsilon\right) \frac{\log t}{\log(t-T)+\log k - \log(\log(C(t-T)+1))} \\ \limsup_{t\to+\infty} E_f(t)\log t \leq \frac{3M_f(T)R^2}{2} + \varepsilon. \end{gathered}$$

We have proved the upper bound (2.4), which implies in particular that $E_f \to 0$ as $t \to +\infty$. Hence, we can deduce from (2.1) that M_f decreases to a nonzero limit: we have proved (2.2) in Theorem 2.1.

2.3.3. *Asymptotic lower bound for energy.* In this subsection, we prove the lower bound (2.5). To bound the energy from below, we need to control the normal derivative of the solution on the boundary. In fact, (2.7) shows that the loss of energy is only due to the evaporation and depends on the normal derivative. More precisely the logarithmic derivative of the energy depends only on the normal derivative and we have

$$\frac{E_f'(t)}{E_f(t)} = R^2 \int_{\partial B_R} \nabla f(t,v)\cdot \frac{v}{|v|}\mathrm{d}\sigma(v).$$

We shall control the normal derivative at time t thanks to a supersolution which vanishes on the boundary at time t. To this purpose, we first need to study an auxiliary function β.

Lemma 2.9. *Let f be the solution of* (1.8), (1.6), (1.5) *and let M_f, E_f be defined by* (1.9). *Consider the unique solution β to the Cauchy problem*

$$\begin{cases} \beta'(t) = 4E_f(t) - 6M_f(t)\beta(t), \\ \beta(0) = \dfrac{2E_{f_0}}{3M_{f_0}}. \end{cases} \tag{2.19}$$

Then β is a positive and nonincreasing function such that

(i) $\frac{2E_f(t)}{3M_f(t)} \leq \beta(t) \leq \frac{2R^2}{3}$,

(ii) $\lim\limits_{t\to+\infty} \beta(t) = 0$.

Proof. It is clear that β is well defined. We know from Lemma 2.5 that M_f does not vanish, so we can consider the function

$$g(t) := \beta(t) - \frac{2E_f(t)}{3M_f(t)}.$$

From (2.1), we have $M_f(t) = M_\infty + \frac{E_f(t)}{R^2}$ and the function

$$\frac{2E_f(t)}{3M_f(t)} = \frac{2R^2}{3}\left(1 - \frac{R^2 M_\infty}{R^2 M_\infty + E_f(t)}\right)$$

is a nonincreasing function of t because E_f is nonincreasing (Lemma 2.5). Thus

$$\beta'(t) \geq 0 \Longrightarrow g'(t) \geq 0.$$

Furthermore, from (2.19) we have

$$g(t) < 0 \Longleftrightarrow \beta'(t) > 0.$$

In particular,

$$g(t) < 0 \Longrightarrow g'(t) > 0.$$

As $g(0) = 0$ we have $g \geq 0$. Indeed, suppose that $t_0 > 0$ is such that $g(t_0) < 0$, define

$$t_1 := \sup\{0 \leq t \leq t_0 : g(t) \geq 0\}$$

We necessarily have $g(t) < 0$ for $t \geq t_1$ and $g(t_1) = 0$ since g is continuous. Thus g is increasing on $[t_1, t_0]$ which contradicts the fact that $g(t_0) < 0 = g(t_1)$. In conclusion we have proved the left inequality in (i) and the fact that β is nonincreasing. We also have the fact that β is positive as E_f is positive (Lemma 2.5 and Lemma 2.4). To end the proof of (i), we remark that, as β is nonincreasing, we have

$$\beta(t) \leq \frac{2E_{f_0}}{3M_{f_0}} \leq \frac{2M_{f_0}R^2}{3M_{f_0}} \leq \frac{2R^2}{3}.$$

Let us now prove (ii). It suffices to prove that β cannot have a positive lower bound. Assume that $\beta(t) \geq \mu > 0$. Then we have

$$\beta'(t) \leq 4E_f(t) - 6M_\infty \mu.$$

As $E_f(t)$ converges to 0 (see (2.4)), there exists $T > 0$ such that for all $t \geq T$, $4E_f(t) \leq 3M_\infty \mu$ and

$$\beta(t) \leq \beta(T) - 3M_\infty \mu (t - T).$$

Since the right-hand side of this inequality becomes negative for large t, this contradicts $\beta(t) \geq \mu > 0$. □

Now we can define our supersolution. Let $T \geq 0$, on D_T we define the Maxwellian

$$\mathcal{M}_T(t,v) := \frac{1}{\beta^{\frac{3}{2}}(t)} e^{-\frac{v^2}{\beta(t)}} - \frac{1}{\beta^{\frac{3}{2}}(T)} e^{-\frac{R^2}{\beta(T)}}, \tag{2.20}$$

where the function $\beta(t)$ is defined in Lemma 2.9.

Lemma 2.10. *Let f be the solution of* (1.8), (1.6), (1.5), *let $T \geq 0$ and let $\mathcal{M}_T$ be defined by* (2.20). *Then there exists a positive constant α independent of T, such that, for all $t \in [0,T]$ and for all $v \in B_R$, we have*

$$f(t,v) \leq \alpha \mathcal{M}_T(t,v). \tag{2.21}$$

In particular, we obtain the following estimate of the normal derivative of f:

$$\forall t \geq 0, \qquad \sup_{v \in \partial B_R} \left| \nabla f(t,v) \cdot \frac{v}{|v|} \right| \leq \frac{2R\alpha}{\beta^{\frac{5}{2}}(t)} e^{-\frac{R^2}{\beta(t)}}. \tag{2.22}$$

Proof. In a first step, we shall find a positive constant α independent of T such that

$$f(0,\cdot) \leq \alpha \mathcal{M}_T(0,\cdot). \tag{2.23}$$

As β is nonincreasing and lower than $\frac{2R^2}{3}$ (Lemma 2.9), the function

$$t \mapsto \frac{1}{\beta^{\frac{3}{2}}(t)} e^{-\frac{R^2}{\beta(t)}}$$

is nonincreasing and we have $\mathcal{M}_T(0,\cdot) \geq \mathcal{M}_0(0,\cdot)$. The function $\mathcal{M}_0(0,\cdot)$ is positive on B_R and vanishes on ∂B_R. We now claim that $\frac{f_0}{\mathcal{M}_0(0,\cdot)}$ is bounded on B_R. Indeed, for $v \in B_R$, we have

$$\frac{f_0(v)}{\mathcal{M}_0(0,v)} = \frac{\int_{|v|}^R \nabla f_0\left(\lambda \frac{v}{|v|}\right) \cdot \frac{v}{|v|} \mathrm{d}\lambda}{\mathcal{M}_0(0,v)}.$$

Defining

$$\mu = \inf_{0 \leq |v| < R} \frac{e^{-\frac{|v|^2}{\beta(0)}} - e^{-\frac{R^2}{\beta(0)}}}{R - |v|},$$

it is clear that $\mu > 0$ and that we have

$$\left| \frac{f_0(v)}{\mathcal{M}_0(0,v)} \right| \leq \frac{\beta^{\frac{3}{2}}(0)}{\mu} \|\nabla f_0\|_\infty,$$

since $f_0 \in \mathcal{C}^1$. Finally, setting

$$\alpha = \max_{v \in \overline{B_R}} \left| \frac{f_0(v)}{\mathcal{M}_0(0,v)} \right| > 0,$$

we obtain (2.23).

In a second step, we shall prove that $\alpha \mathcal{M}_T$ is a supersolution of (1.8), (1.6), (1.5) on D_T. Since we already have (2.23) and $\mathcal{M}_T(t,\cdot) = 0$ on ∂B_R, the only hypothesis that we have to check is

$$\partial_t \mathcal{M}_T - Q_R(\mathcal{M}_T, f) \geq 0.$$

A simple calculation gives

$$\begin{aligned}(\partial_t \mathcal{M}_T - Q_R(\mathcal{M}_T, f))(t,v) &= (2|v|^2 - 3\beta(t))(\beta'(t) + 6M_f(t)\beta(t) - 4E_f(t)) \frac{e^{-\frac{v^2}{\beta(t)}}}{2\beta^{\frac{7}{2}}(t)} \\ &= 0.\end{aligned}$$

Consequently, $\alpha\mathcal{M}_T$ is a supersolution of (1.8), (1.6), (1.5) on D_T. By applying the maximum principle for linear parabolic equation (see [8] for instance), we get (2.21). In particular, since f and $\mathcal{M}_t(t,\cdot)$ vanish on ∂B_R, we have

$$\forall v \in \partial B_R, \qquad 0 \le -\nabla f(t,v)\cdot\frac{v}{|v|} \le -\alpha\nabla\mathcal{M}_t(t,v)\cdot\frac{v}{|v|} = \frac{2R\alpha}{\beta^{\frac{5}{2}}(t)}e^{-\frac{R^2}{\beta(t)}},$$

which gives (2.22). The proof of Lemma 2.10 is complete. ☐

As β converges to 0 (Lemma 2.9), we deduce from (2.22) that

$$\lim_{t\to+\infty}\sup_{v\in\partial B_R}\left|\nabla f(t,v)\cdot\frac{v}{|v|}\right| = 0. \tag{2.24}$$

Estimate (2.22) depends on β but β is itself linked to E_f. We are now able to clarify the behavior of β as $t\to+\infty$.

Lemma 2.11. *Let β be defined by* (2.19)*, we have*

$$\beta(t) \underset{t\to+\infty}{\sim} \frac{2E_f(t)}{3M_\infty}.$$

Proof. We already know from Lemma 2.9 that

$$\beta(t) \ge \frac{2E_f(t)}{3M_f(t)}.$$

Thus, as E_f is positive (Lemma 2.5 and Lemma 2.4), we deduce from (2.2) that

$$\liminf_{t\to+\infty}\frac{\beta(t)}{E_f(t)} \ge \frac{2}{3M_\infty}.$$

Let us show that

$$\limsup_{t\to+\infty}\frac{\beta(t)}{E_f(t)} \le \frac{2}{3M_\infty}. \tag{2.25}$$

Let $0<\varepsilon\le\min\left(\frac{1}{2},\frac{1}{3M_\infty}\right)$, from (2.24) we know that there exists $T_1\ge 0$ such that, for all $t\ge T_1$,

$$\sup_{v\in\partial B_R}\left|\nabla f(t,v)\cdot\frac{v}{|v|}\right| \le \frac{9M_\infty^2\varepsilon}{16\pi R^4}.$$

For $t\ge T_1$, define

$$g(t) := \frac{\beta(t)}{E_f(t)}.$$

Using (2.19) and (2.7) we have

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \frac{\beta'(t)}{\beta(t)} - \frac{E_f'(t)}{E_f(t)}\\ &= \frac{4E_f(t)}{\beta(t)} - 6M_f(t) - R^2\int_{\partial B_R}\nabla f(t,v)\cdot\frac{v}{|v|}\mathrm{d}\sigma(v).\end{aligned}$$

If $g(t)\ge\frac{2}{3M_\infty}+\varepsilon$ then

$$\frac{4E_f(t)}{\beta(t)} \le \frac{6M_\infty}{1+\frac{3M_\infty\varepsilon}{2}} \le 6M_\infty\left(1-\frac{3M_\infty\varepsilon}{4}\right),$$

and since $M_\infty\le M_f(t)$ (Lemma 2.5) we have

$$\frac{g'(t)}{g(t)} \le -\frac{9M_\infty^2\varepsilon}{4}.$$

It is clear then, that there exists $T_2 \geq T_1$ such that $g(T_2) \leq \frac{2}{3M_\infty} + \varepsilon$ otherwise we have

$$\forall t \geq T_1 \quad \frac{2}{3M_\infty} + \varepsilon \leq g(t) \leq g(T_1) e^{-\frac{9M_\infty^2 \varepsilon}{4} t},$$

a contradiction. Now, define for $t \geq T_2$

$$h(t) := \frac{2}{3M_\infty} + \varepsilon - g(t).$$

We have proved that $h(t) < 0 \Longrightarrow h'(t) > 0$ and, by definition, $h(T_2) \geq 0$. We can conclude as in the proof of Lemma 2.9 that $h \geq 0$. Finally, for all $t \geq T_2$, we have $g(t) \leq \frac{2}{3M_\infty} + \varepsilon$ and

$$\limsup_{t \to +\infty} \frac{\beta(t)}{E_f(t)} \leq \frac{2}{3M_\infty} + \varepsilon.$$

This proves (2.25) and the proof of the lemma is complete. □

We are now in position to prove (2.5). From (2.7) and (2.22), we deduce

$$\forall t \geq 0 \quad \frac{E_f'(t)}{E_f(t)} \geq -\frac{8\pi R^5 \alpha}{\beta^{\frac{5}{2}}(t)} e^{-\frac{R^2}{\beta(t)}}.$$

Let $0 < \varepsilon < \frac{3M_\infty R^2}{2}$, we have then

$$-\frac{E_f'(t)}{E_f^2(t)} e^{\frac{1}{E_f(t)}\left(\frac{3M_\infty R^2}{2} - \varepsilon\right)} \leq \frac{8\pi R^5 \alpha e^{R^2\left(\frac{3M_\infty}{2E_f(t)} - \frac{1}{\beta(t)}\right) - \frac{\varepsilon}{E_f(t)}}}{\beta^{\frac{5}{2}}(t) E_f(t)}.$$

With Lemma 2.11, the right-hand side of this inequality tends to 0+ when t grows to infinity and there exists a positive constant C such that, for all $t \geq 0$,

$$-\left(\frac{3M_\infty R^2}{2} - \varepsilon\right) \frac{E_f'(t)}{E_f^2(t)} e^{\frac{1}{E_f(t)}\left(\frac{3M_\infty R^2}{2} - \varepsilon\right)} \leq C,$$

$$e^{\frac{1}{E_f(t)}\left(\frac{3M_\infty R^2}{2} - \varepsilon\right)} - e^{\frac{1}{E_{f0}}\left(\frac{3M_\infty R^2}{2} - \varepsilon\right)} \leq Ct,$$

$$E_f(t) \log t \geq \frac{\left(\frac{3M_\infty R^2}{2} - \varepsilon\right) \log t}{\log\left(Ct + e^{\frac{1}{E_{f0}}\left(\frac{3M_\infty R^2}{2} - \varepsilon\right)}\right)},$$

$$\liminf_{t \to +\infty} E_f(t) \log t \geq \frac{3M_\infty R^2}{2} - \varepsilon.$$

The proof of (2.5) is complete.

2.3.4. *End of the proof.* We are ready to end the proof of Theorem 2.1. The existence and uniqueness of a smooth solution to (1.8), (1.6), (1.5) have been proved in Section 2.3.1. The conservation law (2.1) and the monotonicity of M_f and E_f are given in Lemma 2.5. In Section 2.3.2, we have proved the upper bound (2.4) and the non-zero mass limit (2.2). In Section 2.3.3, we have proved the lower bound (2.5). These two bounds imply the logarithmic law (2.3). It remains to identify the limit of $f(t,.)$.

The convergence of $f(t,.)$ to $M_\infty \delta_{v=0}$ in the sense of narrow convergence of measures is a standard consequence of the following properties. First, for all $t \geq 0$, $f(t,.)$ is a bounded nonnegative function of $\mathbb{L}^1(B_R)$, whose mass goes to $M_\infty > 0$. Second, the energy $\int_{B_R} f(v) v^2 \mathrm{d}v$ converges to zero as $t \to +\infty$.

3. Numerical results

Our goal in this section is to construct a suitable numerical scheme to solve (1.8) and provide simulations to illustrate our theoretical results: conservation law (2.1) and slow evaporation (2.3). We emphasize on the particular attention that should be made concerning the conservation law in the numerical scheme, otherwise the numerical solution would dissipate to 0 as any solution to the heat equation (with zero boundary condition) should do. This conservation should therefore be exactly satisfied by the scheme to avoid this wrong behavior. Conservative schemes for axisymmetric Fokker-Planck-Landau operator without the Dirichlet boundary condition are discussed in [13]. We consider only isotropic (radial) distribution functions so that the problem to simulate is a 1D problem. We construct a scheme that satisfies a discretized version of the conservation law (2.1) and ensures the positivity of the distribution function under a CFL condition. The velocity domain $[-R, R]$ is first reduced to $[0, R]$ thanks to the parity property that our scheme inherits from the continuous model. A convergence to a numerical Dirac distribution is shown and the numerical results are coherent with the theoretical behaviors. However, we cannot reach an acceptable numerical stationary state because the convergence to the equilibrium is very slow. It is not surprising since energy governs the dynamics and it decreases very slowly. Anyway, it is clear from simulations that the decrease of energy is not exponential.

3.1. Numerical scheme.

3.1.1. *Notations and definitions.* For the numerical scheme we take $R = 1$. To avoid the calculation of the divergence in spherical coordinates at 0, we choose a shifted grid. The velocity step is defined by

$$\Delta v := \frac{2}{2N+1},$$

with $N \in \mathbb{N}^*$. We work with the following set of indices:

$$X := \left[-\frac{2N+1}{2}, \frac{2N+1}{2}\right] \cap \frac{1}{2}\mathbb{Z}.$$

The grid is given by

$$v_k := k\Delta v \quad k \in X, \qquad t_n := n\Delta t \quad n \in \mathbb{N}.$$

To preserve the parity, we use the following discrete derivative with respect to velocity

$$D_k g_k^n := \begin{cases} \dfrac{g_{k+\frac{1}{2}}^n - g_{k-\frac{1}{2}}^n}{\Delta v} & \text{if } -N \leq k \leq N \\ -\dfrac{g_N^n}{\Delta v} & \text{if } k = N + \dfrac{1}{2} \\ \dfrac{g_{-N}^n}{\Delta v} & \text{if } k = -N - \dfrac{1}{2} \end{cases}$$

We define two different discrete integrals, one uses integer indices, the other uses non-integer indices:

$$I(g_k^n) := \sum_{k=-N+1}^{N-1} g_k^n \Delta v, \qquad J(g_k^n) := \sum_{k=-N}^{N+1} g_{k-\frac{1}{2}}^n \Delta v.$$

For $k \in X \setminus \mathbb{Z}$, we define f_k^n an approximation of $f(t_n, v_k)$ where f is the solution of our problem. For convenience we define f_k^n for $k \in X \cap \mathbb{Z}$ by

$$f_k^n := \begin{cases} f_{k+\frac{1}{2}}^n & \text{if } k > 0 \\ f_{k-\frac{1}{2}}^n & \text{if } k < 0 \\ \dfrac{f_{\frac{1}{2}}^n + f_{-\frac{1}{2}}^n}{2} & \text{if } k = 0 \end{cases} \qquad k \in X \cap \mathbb{Z} \tag{3.1}$$

The purpose of this definition is to allow an easy-handling formulation of the scheme. In fact, these values are artificial and this definition simply implies that the scheme is excentered coherently with the transport term. For $n = 0$ we use the initial condition

$$f_k^0 := f_0(v_k) \quad k \in X \setminus \mathbb{Z}. \tag{3.2}$$

For all $n \in \mathbb{N}$, we have the boundary condition

$$f_{-N-\frac{1}{2}}^n = f_{N+\frac{1}{2}}^n = 0. \tag{3.3}$$

The mass and the energy of the numerical solution are defined by

$$M_n := 2\pi J(v_k^2 f_k^n), \qquad E_n := 2\pi J(v_k^4 f_k^n). \tag{3.4}$$

However, in order to construct a conservative scheme, we introduce a different discretization of the mass and the energy for the coefficients of the equation. We set

$$\tilde{M}_n := -\frac{2\pi}{3} I(v_k^3 D_k f_k^n), \qquad \tilde{E}_n := 2\pi I(v_k^4 f_k^n). \tag{3.5}$$

The definition of $\tilde{M}_n$ comes from the following reformulation

$$2\pi \int_{-1}^{1} f(t,v) v^2 \mathrm{d}v = \frac{2\pi}{3} \int_{-1}^{1} f(t,v) \frac{\mathrm{d}}{\mathrm{d}v}(v^3) \mathrm{d}v = -\frac{2\pi}{3} \int_{-1}^{1} \partial_v f(t,v) v^3 \mathrm{d}v.$$

Let us now define our numerical scheme. The solution at step $n+1$ is defined from the solution at step n according to

$$f_k^{n+1} = f_k^n + \frac{\Delta t}{v_k^2} D_k (v_k^2 \tilde{E}_n D_k f_k^n + 3 v_k^3 \tilde{M}_n f_k^n). \tag{3.6}$$

3.1.2. *Properties of the numerical scheme.* The above defined numerical scheme satisfies the following properties.

Proposition 3.1. *Let f_0 be a continuous nonnegative and even function on $[-1,1]$ such that $f(-1) = f(1) = 0$. Let $N \geq 3$ and f_k^n defined for $(n,k) \in \mathbb{N} \times X$ by* (3.6)*,* (3.2)*,* (3.3) *and* (3.1)*. Suppose that*

$$\Delta t \leq \frac{(\Delta v)^2}{9 M_0}. \tag{3.7}$$

Then the following properties hold:

(i) $\forall n \in \mathbb{N}, \forall k \in X \quad f_{-k}^n = f_k^n$,
(ii) $\forall n \in \mathbb{N}, \forall k \in X \quad f_k^n \geq 0$,
(iii) $\forall n \in \mathbb{N} \quad M_{n+1} \leq M_n \leq M_0$,
(iv) $\forall n \in \mathbb{N} \quad E_{n+1} \leq E_n \leq E_0$,
(v) $\forall n \in \mathbb{N} \quad (1-\Delta v)^2 M_n - E_n = (1-\Delta v)^2 M_0 - E_0$.

Remark 3.2. Properties (ii) and (iii) give the $\mathbb{L}^1$-stability of the scheme.

Remark 3.3. As $R = 1$, $(1-\Delta v)^2 \simeq R^2$, (v) is a discrete formulation of (2.1).

Proof. The proof of (i) is a simple verification. Let us study the evolution of mass and energy. For convenience we denote

$$g_k^n := v_k^2 \tilde{E}_n D_k f_k^n + 3 v_k^3 \tilde{M}_n f_k^n,$$

where $\tilde{M}_n$ and $\tilde{E}_n$ are defined by (3.5). Let $n \in \mathbb{N}$. We have

$$\begin{aligned} M_{n+1} - M_n &= 2\pi J(v_k^2 (f_k^{n+1} - f_k^n)) \\ &= 2\pi \Delta t \Delta v \sum_{k=-N+1}^{N} D_k g_{k-\frac{1}{2}}^n \\ &= 2\pi \Delta t \sum_{k=-N+1}^{N} (g_k^n - g_{k-1}^n) \\ &= 2\pi \Delta t (g_N^n - g_{-N}^n) \\ &= 4\pi \Delta t g_N^n, \end{aligned}$$

and, finally,

$$M_{n+1} - M_n = -4\pi \frac{\Delta t}{\Delta v} v_N^2 \tilde{E}_n f_{N-\frac{1}{2}}^n. \tag{3.8}$$

For the energy we have

$$\begin{aligned} E_{n+1} - E_n &= 2\pi J(v_k^4 (f_k^{n+1} - f_k^n)) \\ &= 2\pi \Delta t \Delta v \sum_{k=-N+1}^{N} v_{k-\frac{1}{2}}^2 D_k g_{k-\frac{1}{2}}^n \\ &= 2\pi \Delta t (v_{N-\frac{1}{2}}^2 g_N^n - v_{-N+\frac{1}{2}}^2 g_{-N}^n) + 2\pi \Delta t \sum_{k=-N+1}^{N-1} (v_{k-\frac{1}{2}}^2 - v_{k+\frac{1}{2}}^2) g_k^n \\ &= 4\pi \Delta t v_{N-\frac{1}{2}}^2 g_N^n - 4\pi \Delta t \Delta v \sum_{k=-N+1}^{N-1} v_k g_k^n. \end{aligned}$$

We have

$$\begin{aligned} \sum_{k=-N+1}^{N-1} v_k g_k^n &= \sum_{k=-N+1}^{N-1} v_k^3 \tilde{E}_n D_k f_k^n + 3 v_k^4 \tilde{M}_n f_k^n \\ &= 2\pi \Delta v \sum_{k=-N+1}^{N-1} \sum_{k_*=-N+1}^{N-1} (v_k^3 v_{k_*}^4 f_{k_*}^n D_k f_k^n - v_{k_*}^3 v_k^4 f_k^n D_k f_{k_*}^n) \\ &= 0, \end{aligned}$$

since it is the sum of the coefficients of an antisymmetric matrix. Thus

$$E_{n+1} - E_n = -4\pi \frac{\Delta t}{\Delta v} v_N^2 (1 - \Delta v)^2 \tilde{E}_n f_{N-\frac{1}{2}}^n. \tag{3.9}$$

From (3.8) and (3.9) we obtain (v). Furthermore, if (ii) is verified up to time n then we have (iii) and (iv) at time n. To finish the proof, we prove (ii) at time $n+1$. For this purpose we bound $\tilde{M}_n$ and $\tilde{E}_n$ with M_0. First we calculate the difference between $\tilde{M}_n$ and M_n:

$$\begin{aligned}\tilde{M}_n &= -\frac{2\pi}{3}\sum_{k=-N+1}^{N-1} v_k^3(f^n_{k+\frac12} - f^n_{k-\frac12})\\ &= -\frac{2\pi}{3}(-v_{-N}^3 f^n_{-N+\frac12} + v_N^3 f^n_{N-\frac12}) - \frac{2\pi}{3}\sum_{k=-N+1}^{N}(v_{k-1}^3 - v_k^3)f^n_{k-\frac12}\\ &= -\frac{4\pi}{3}v_N^3 f^n_{N-\frac12} - \frac{2\pi}{3}\sum_{k=-N}^{N+1}\left(-3v_{k-\frac12}^2\Delta v - \frac14(\Delta v)^3\right)f^n_{k-\frac12}\\ &= M_n - \frac{4\pi}{3}v_N^3 f^n_{N-\frac12} + \frac{\pi}{6}\sum_{k=-N}^{N+1}(\Delta v)^3 f^n_{k-\frac12}.\end{aligned}$$

Moreover, as $N \geq 3$, we have

$$\frac{\pi}{6}\sum_{k=-N}^{N+1}(\Delta v)^3 f^n_{k-\frac12} \leq \frac{\Delta v}{3}M_n \leq \frac{2}{21}M_n,$$

and

$$\frac{4\pi}{3}v_N^3 f^n_{N-\frac12} \leq \frac23\frac{N^2}{\left(N-\frac12\right)^2}2\pi v_{N-\frac12}^2 f^n_{N-\frac12} \leq \frac23\frac{N^2}{\left(N-\frac12\right)^2}M_n \leq M_n.$$

Finally

$$0 \leq \tilde{M}_n \leq \frac{23}{21}M_0.$$

Analogously, it is easy to see that with $N \geq 3$

$$0 \leq \tilde{E}_n \leq \frac{61}{49}M_0.$$

From (3.6), we deduce (ii) under the condition that, for all $k \in X$,

$$1 - \tilde{E}_n\frac{v_{k+\frac12}^2 + v_{k-\frac12}^2}{v_k^2}\frac{\Delta t}{(\Delta v)^2} - 3\tilde{M}_n\frac{v_{k-\frac12}}{v_k^2}\frac{\Delta t}{\Delta v} \geq 0.$$

For that, the following condition is sufficient:

$$\Delta t \leq \frac{(\Delta v)^2}{9M_0}.$$

□

The conservation properties imply that the numerical solution converges, when n tends to infinity and for fixed values of Δv and Δt, to a discrete Dirac whereas simpler nonconservative scheme converges to 0. More precisely we have the following result.

Proposition 3.4. *Let f_0 be a continuous nonnegative and even function on $[-1,1]$ such that $f(-1) = f(1) = 0$. Let $N \geq 3$ and f_k^n defined for $(n,k) \in \mathbb{N}\times X$ by* (3.6), (3.2), (3.3) *and* (3.1). *Assume that $\Delta t \leq \frac{(\Delta v)^2}{9M_0}$ and that*

$$(1-\Delta v)^2 M_0 - E_0 > 0.$$

Then,

(i) *for* $k \notin \{-\frac{1}{2}, 0, \frac{1}{2}\}$, *we have*

$$\lim_{n\to+\infty} f_k^n = 0$$

and

$$\lim_{n\to+\infty} f_{\pm\frac{1}{2}}^n = \lim_{n\to+\infty} f_0^n = \frac{(1-\Delta v)^2 M_0 - E_0}{(1-\Delta v)^2 \pi (\Delta v)^3 - \frac{\pi}{4}(\Delta v)^5} > 0.$$

(ii) *Moreover, we have*

$$\lim_{n\to+\infty} M_n = \frac{(1-\Delta v)^2 M_0 - E_0}{(1-\Delta v)^2 - \frac{1}{4}(\Delta v)^2},$$

$$\lim_{n\to+\infty} E_n = \frac{1}{4}(\Delta v)^2 \frac{(1-\Delta v)^2 M_0 - E_0}{(1-\Delta v)^2 - \frac{1}{4}(\Delta v)^2},$$

$$\lim_{n\to+\infty} \tilde{M}_n = \frac{4}{3}\frac{(1-\Delta v)^2 M_0 - E_0}{(1-\Delta v)^2 - \frac{1}{4}(\Delta v)^2},$$

$$\lim_{n\to+\infty} \tilde{E}_n = 0,$$

where M_n, E_n, $\tilde{M}_n$ *and* $\tilde{E}_n$ *are defined by (3.4) and (3.5).*

Remark 3.5. For any non zero initial condition that vanishes on the boundary, we have $M_0 > E_0$. Therefore, if Δv is chosen small enough, the assumption $(1-\Delta v)^2 M_0 - E_0 > 0$ is satisfied.

Proof. From Items (ii) and (iii) of Proposition 3.1, we know that, up to a subsequence, the values f_k^n for $k \in X \setminus \mathbb{Z}$ and thus for all $k \in X$ converge, i.e. we can choose $\varphi : \mathbb{N} \to \mathbb{N}$ increasing such that

$$\forall k \in X \quad \lim_{n\to+\infty} f_k^{\varphi(n)} =: f_k^\infty,$$

where the limits f_k^∞ depend on φ. Therefore

$$\lim_{n\to+\infty} \tilde{M}_{\varphi(n)} =: \tilde{M}_\infty, \lim_{n\to+\infty} \tilde{E}_{\varphi(n)} =: \tilde{E}_\infty,$$

as limits of linear combinations of the f_k^n. Besides, using (3.6) and (3.1), $f_k^{\varphi(n)+1}$ can be expressed in function of $f_k^{\varphi(n)}$, $\tilde{E}_{\varphi(n)}$ and $\tilde{M}_{\varphi(n)}$. Thus, it converges to a limit $f_k^{\infty,1}$. Let us denote by $\tilde{E}_{\infty,1}$ and $\tilde{M}_{\infty,1}$ the limits of $\tilde{E}_{\varphi(n)+1}$ and $\tilde{M}_{\varphi(n)+1}$ respectively. In the same way, we can define, by induction on $p \in \mathbb{N}$, $f_k^{\infty,p}$, $\tilde{E}_{\infty,p}$ and $\tilde{M}_{\infty,p}$. From (3.6), we have, for all $p \in \mathbb{N}$,

$$\forall k \in X \setminus \mathbb{Z} \quad \frac{f_k^{\infty,p+1} - f_k^{\infty,p}}{\Delta t} = \frac{1}{v_k^2} D_k(v_k^2 \tilde{E}_{\infty,p} D_k f_k^{\infty,p} + 3 v_k^3 \tilde{M}_{\infty,p} f_k^{\infty,p}). \tag{3.10}$$

From (iv) and (ii) of Theorem 3.1 we have

$$\lim_{n\to+\infty} E_n =: E_\infty.$$

Then, from (3.9) we get

$$\forall p \in \mathbb{N} \quad \tilde{E}_{\infty,p} f_{N-\frac{1}{2}}^{\infty,p} = 0.$$

Assume that, for every $p \in \mathbb{N}$, we have $\tilde{E}_{\infty,p} \neq 0$. Then, for $q \geq 1$ and using (3.10) with $p = q-1$ and $k = N - \frac{1}{2}$, we obtain $f_{N-\frac{3}{2}}^{\infty,q-1} = f_{N-\frac{1}{2}}^{\infty,q-1} = 0$. By finite induction, for $s \leq N-1$, we deduce that if $q \geq s$

$$f_{N-\frac{1}{2}-s}^{\infty,q-s} = f_{N-\frac{1}{2}-s+1}^{\infty,q-s} = \cdots = f_{N-\frac{1}{2}}^{\infty,q-s} = 0.$$

Therefore, for $q \geq 0$, we have $f_k^{\infty,q} = 0$ for all $k \in X$ and then $\tilde{E}_{\infty,q} = 0$, which contradicts our hypothesis. Thus, there exists $p_0 \in \mathbb{N}$ such that $\tilde{E}_{\infty,p_0} = 0$. As

$$0 = \tilde{E}_{\infty,p_0} = 2\pi I(v_k^4 f_k^{\infty,p_0}),$$

we deduce from (ii) that

$$\forall k \in X \setminus \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\} \quad f_k^{\infty,p_0} = 0.$$

Consider the smallest p_0 for which $\tilde{E}_{\infty,p_0} = 0$. Suppose that $p_0 \geq 1$, then $f_{N-\frac{3}{2}}^{\infty,p_0} = f_{N-\frac{1}{2}}^{\infty,p_0} = f_{N-\frac{1}{2}}^{\infty,p_0-1} = 0$ and from (3.10) $f_{N-\frac{3}{2}}^{\infty,p_0-1} = 0$. A simple finite induction on $s \leq N-1$ gives

$$0 = f_{N-\frac{1}{2}}^{\infty,p_0-1} = f_{N-\frac{3}{2}}^{\infty,p_0-1} = \cdots = f_{N-\frac{1}{2}-s}^{\infty,p_0-1} = \cdots = f_{\frac{1}{2}}^{\infty,p_0-1} = 0,$$

and thus $\tilde{E}_{\infty,p_0-1} = 0$ which contradicts the definition of p_0. Finally, $p_0 = 0$ and

$$\lim_{n\to+\infty} \tilde{E}_n = 0,$$

$$\forall k \in X \setminus \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\} \quad \lim_{n\to+\infty} f_k^n = 0,$$

$$E_\infty = \frac{\pi}{4}(\Delta v)^5 f_{\frac{1}{2}}^\infty,$$

$$\lim_{n\to+\infty} M_n = \pi(\Delta v)^3 f_{\frac{1}{2}}^\infty =: M_\infty.$$

$$\lim_{n\to+\infty} \tilde{M}_n = \frac{4\pi}{3}(\Delta v)^3 f_{\frac{1}{2}}^\infty.$$

To finish the proof, we use the discrete conservation law (v) of Theorem 3.1 that gives:

$$(1-\Delta v)^2 M_\infty - E_\infty = (1-\Delta v)^2 M_0 - E_0.$$

□

3.2. **Simulations.** On Figures 1, 2 and 3, one observes that the shape of the initial condition has poor influence on the behavior of the solution after a short time. Indeed, rapidly the solution looks like a Maxwellian, which illustrates numerically the self-similar behavior proved in the future work [2].

Table 1 shows the different values of the distribution function at $v = \frac{1}{9}$ and illustrates the convergence of the scheme as the grid steps tend to zero. The velocity step is divided by 3 instead of 2, because of the shifted grid chosen to avoid the zero value. The velocity step is taken so that $\frac{1}{9}$ is always a value in the velocity grid and the time step is chosen according to the CFL condition. For this calculation we used the initial condition of Figure 1.

Δv	Δt	$f(t,v)$	CPU
$\frac{2}{9}$	$\frac{4}{729}$	2.49734	0.04s
$\frac{2}{27}$	$\frac{4}{6561}$	3.11357	0.012s
$\frac{2}{81}$	$\frac{4}{59049}$	2.92593	0.026s
$\frac{2}{243}$	$\frac{4}{531441}$	2.82668	0.144s
$\frac{2}{729}$	$\frac{4}{4782969}$	2.79005	1.451s
$\frac{2}{2187}$	$\frac{4}{43046721}$	2.77747	44.525s

TABLE 1. Test of scheme convergence, at time $t = 4$ for $|v| = \frac{1}{9}$.

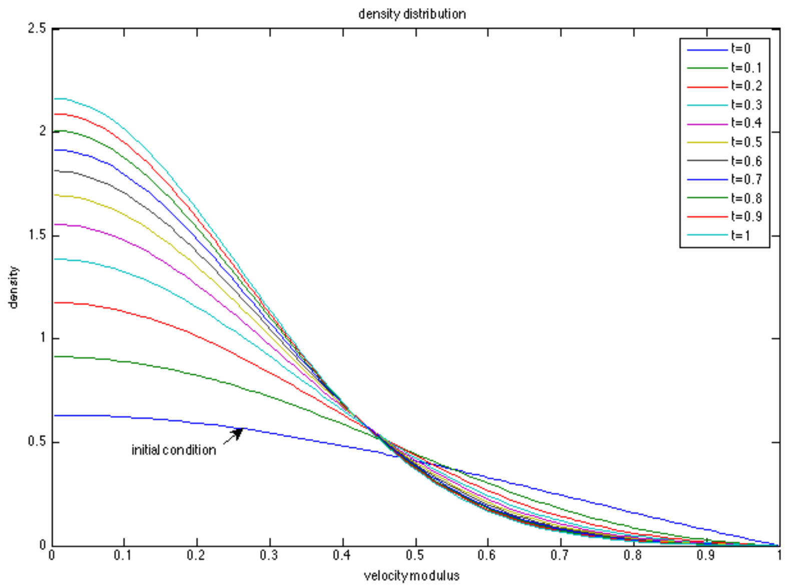

FIGURE 1. Density distribution at different times for a shifted Maxwellian initial condition: $f_0(v) = e^{-v^2} - 1/e$

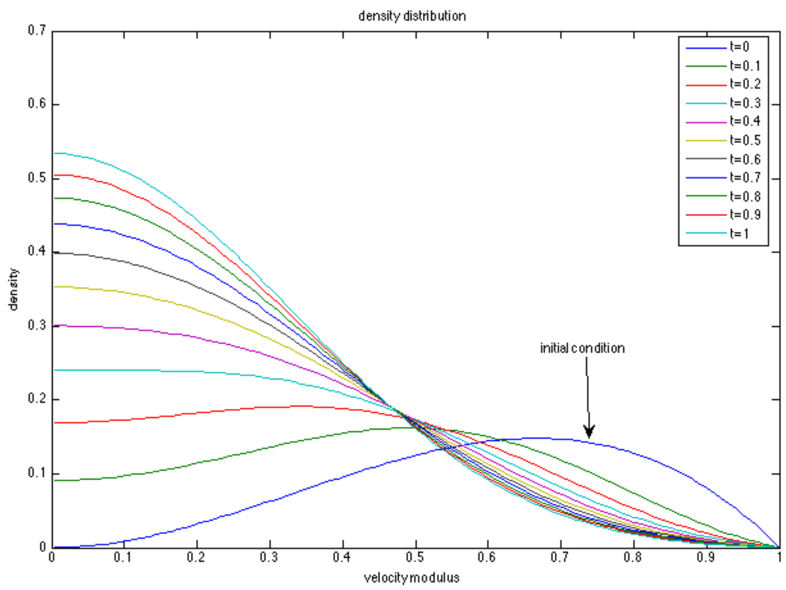

FIGURE 2. Density distribution at different times for an off-centered bump initial condition: $f_0(v) = |v|^2 - |v|^3$

Since the long time convergence is very slow, we are not able to check numerically that the product $E_f(t)\log t$ converges to the right constant $\frac{3M_\infty R^2}{2}$ which is about 0.75687 for the initial condition considered. Nevertheless, we observed that for relatively large time, the profile of $E_f(t)\log t$ closely follows that of the theoretical

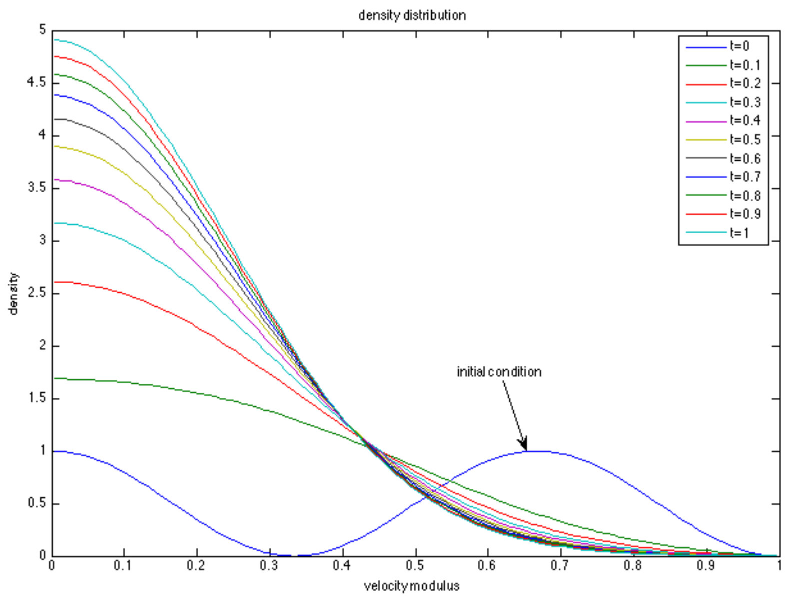

FIGURE 3. Density distribution at different times for a double bump initial condition with a zero normal derivative on the boundary: $f_0(v) = \frac{\cos(3\pi|v|)+1}{2}$

lower bound found in Section 2.3.3:

$$t \mapsto \frac{\left(\frac{3M_\infty R^2}{2} - \varepsilon\right)\log t}{\log\left(Ct + e^{\frac{1}{E_{f_0}}\left(\frac{3M_\infty R^2}{2} - \varepsilon\right)}\right)}$$

On Figure 4, we have represented the product $E_f(t)\log(t)$ and the theoretical lower bound. We have chosen $\varepsilon = 0.01$ and determined numerically the constant α of Lemma 2.10 to compute a value of C as the maximum of

$$\frac{8\pi R^5 \alpha e^{R^2\left(\frac{3M_\infty}{2E_f(t)} - \frac{1}{\beta(t)}\right) - \frac{\varepsilon}{E_f(t)}}}{\beta^{\frac{5}{2}}(t)E_f(t)}$$

for $t \in [0, 100]$. We found $C \simeq 7208.3$.

Let us now illustrate numerically that the distribution function converges to the Dirac mass. Figure 5 shows the numerical solution, when $\Delta v = \frac{2}{7}$ and $\Delta t = \frac{4}{441}$, with the initial condition of Figure 1, up to time $T = 10^6$. In Table 2, we give the calculated values compared with the theoretical limits given by Proposition 3.4 for $\Delta v = \frac{2}{7}$ and $\Delta t = \frac{4}{441}$ at time $T = 10^9$.

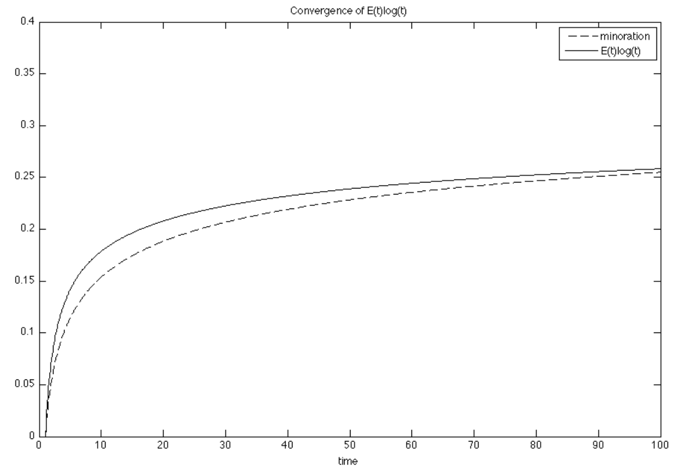

FIGURE 4. Evolution of $E_f(t)\log t$

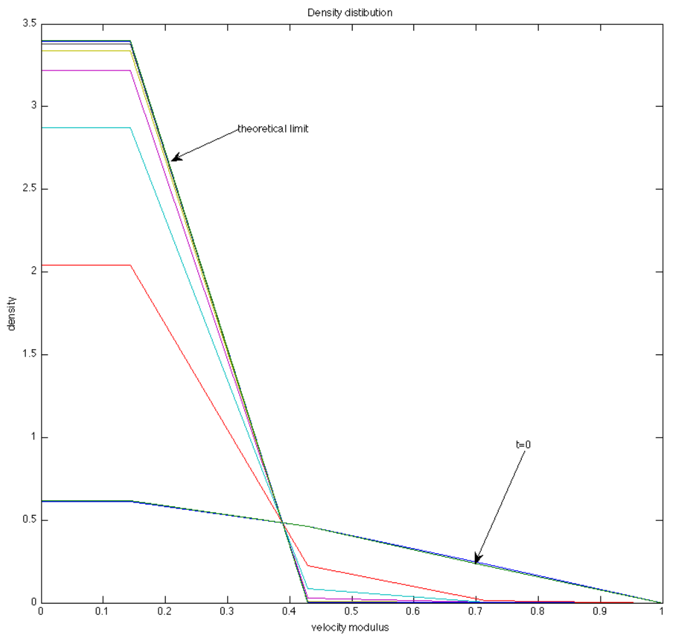

FIGURE 5. Density distribution at time $t = 0, 1, 10, 100, \ldots, 10^6$ for $f_0(v) = e^{v^2} - \frac{1}{e}$, $\Delta v = \frac{2}{7}$ and $\Delta t = \frac{4}{441}$.

	numerical value	theoretical limit	relative error
$f^T_{\frac{1}{2}}$	3.397701	3.397888	$5.5.10^{-5}$
$f^T_{\frac{3}{2}}$	$3.108070.10^{-5}$	0	
$f^T_{\frac{5}{2}}$	$1.421556.10^{-10}$	0	
E_T	$5.084596.10^{-3}$	$5.081111.10^{-3}$	$6.9.10^{-4}$
$\tilde{E}_T$	$7.436903.10^{-7}$	0	

TABLE 2. Numerical values at time $T = 10^9$ with $\Delta v = \frac{2}{7}$, $\Delta t = \frac{4}{441}$.

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