The Child-Langmuir asymptotics applied to a bipolar diode

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Abstract

We apply the Child-Langmuir asymptotics of the Vlasov-Poisson system to the case of a bipolar diode, i.e. a vacuum diode where two species of particles of opposite electric charge are flowing. This leads to a simplified model which, if at least one of the two injected currents is not too large, has a unique solution. Moreover, in that case, the currents flowing inside the diode are limited by the so-called bipolar Child-Langmuir currents. In the case of large currents, other solutions may appear, and formation of virtual electrodes may occur inside the diode.

1. Introduction

In this paper we investigate the Child-Langmuir asymptotics applied to a plane diode. Both electrodes emit particles of opposite electric charges, ions and electrons. Such a diode is called a *bipolar diode*. This situation occurs for instance in the modelling of ion diodes [15]: if an ion beam, emitted by an anode, is accelerated by a strong applied voltage, a secondary electronic emission from the cathode usually appears, and a bipolar flow takes place inside the diode.

Let us first briefly recall the principles of this asymptotics. The modelling and the simulation of numerous devices involving charged particle transport require the study of the coupling between the particle transport equations and the electromagnetic fields equations. This coupling is expressed by the Vlasov-Poisson or the Vlasov-Maxwell systems. Appropriate boundary conditions should be provided to describe real physical cases. The stationary problems associated to these systems were recently studied by Greengard-Raviart [12] and Poupaud [17].

Since the early 30's, Child, Langmuir and Compton [13] have stated that the current, which can flow through a vacuum plane diode, cannot exceed a maximal value. This result was established for electrons emitted by a cathode: a space-charge layer takes place in the vicinity of the cathode, and an electric potential barrier limits the emission of electrons inside the diode. The maximal current J_1^{CL} depends on the applied potential Φ_L and on the distance L between the electrodes. It has the form

$$J_1^{\mathrm{CL}} = \frac{4}{9}\varepsilon_0\sqrt{\frac{2e}{m}}\frac{\Phi_L^{3/2}}{L^2}$$

where ε_0 is the vacuum permittivity and e, m are the electron charge and mass.

A mathematical framework to describe this space-charge limitation regime was proposed by Degond and Raviart [10, 11], where a singular perturbation analysis, called the Child-Langmuir asymptotics, was introduced. The ratio ε^2, of the average energy of the emitted particles over the external bias, is assumed to be small, whereas the injected current is finite. This leads to a singularly perturbed boundary condition for the Vlasov equation. It was used to study an electron flow inside a plane diode in [10]. The same asymptotics has also been applied to several other cases: when the space dimension is greater than one and the diode has a certain symmetry [9], when there is a background of positive ions [2], with a collision operator [3, 4], in the quantum case [5], for the Vlasov-Maxwell system [6, 7] or in a non-stationary case [1]. A general presentation of this asymptotics is presented in [8].

In this paper, we treat the case of several species of particles, for which the bipolar diode is a model problem. We consider the nonrelativistic Vlasov-Poisson system. A relativistic treatment of the particles, formally done in [16], would be of the same technical difficulty [7]. Since the convexity of the potential is not guaranteed, it is not clear, *a priori*, that a space-charge limitation of the current occurs. It is interesting to compare our problem with the one treated by Ben Abdallah in [2], when the density of the ions is replaced by a given doping density. Let us briefly recall the results obtained in that case: for a small doping density, it was proved that the classical results of [13] can be extended. In the case of a large doping density, the phase portrait of the diode can be strongly modified and parasitic solutions may appear, where some electrons are trapped inside the diode. Hence, for the bipolar diode, we may also expect some changes from the classical results [13], [10], even if the ion density is not a given datum.

The outline of the paper is the following. In section 2., we introduce the model and describe our main results. In section 3., we prove that, as the small parameter ε tends to zero, the model converges to a limit model. Section 4. is finally devoted to the study of this reduced model in two different cases, of small and large currents, the most complete result concerning the case of small currents.

2. The singular perturbation problem and description of the results

2.1. The model and an existence theorem

We consider a plane diode, which consists of a pair of electrodes: a cathode located at $X = 0$, at the potential $\Phi(0) = 0$, and an anode at $X = L$, $\Phi(L) = \Phi_L$. The cathode emits electrons of charge $-e$ and mass m_e and the anode emits ions of charge $+e$ and mass m_i.

For $(X, V) \in [0, L] \times \mathbb{R}$, $F_e(X, V)$ and $F_i(X, V)$ denote respectively the distribution functions of the electrons and of the ions. For $V > 0$, let $G_e(V)$ be the distribution of the injected electrons at the cathode and, for $V < 0$, let $G_i(V)$ be the distribution of the injected ions at the anode.

The stationary Vlasov-Poisson system writes

$$\begin{cases} V\dfrac{\partial F_e}{\partial X} + \dfrac{e}{m_e}\dfrac{d\Phi}{dX}\dfrac{\partial F_e}{\partial V} = 0, & (X,V) \in [0,L] \times \mathbb{R}, \quad (2.1) \\[2ex] V\dfrac{\partial F_i}{\partial X} - \dfrac{e}{m_i}\dfrac{d\Phi}{dX}\dfrac{\partial F_i}{\partial V} = 0, & (X,V) \in [0,L] \times \mathbb{R}, \quad (2.2) \\[2ex] \dfrac{d^2\Phi}{dX^2} = \dfrac{e}{\varepsilon_0}(N_e - N_i), & X \in [0,L], \quad (2.3) \end{cases}$$

where

$$N_e(X) = \int_{\mathbb{R}} F_e(X,V)dV, \qquad N_i(X) = \int_{\mathbb{R}} F_i(X,V)dV. \tag{2.4}$$

The boundary conditions corresponding to the emissions of particles and to the applied potential are

$$\begin{cases} F_e(0,V) = G_e(V), & V > 0, \quad (2.5) \\ F_e(L,V) = 0, & V < 0, \quad (2.6) \\[2ex] F_i(0,V) = 0, & V > 0, \quad (2.7) \\ F_i(L,V) = G_i(V), & V < 0, \quad (2.8) \\[2ex] \Phi(0) = 0, \qquad \Phi(L) = \Phi_L. & \quad (2.9) \end{cases}$$

We consider the macroscopic quantities associated to the entering distributions G_e and G_i:

$$N_e^G = \int_0^{+\infty} G_e(V)dV, \qquad N_i^G = \int_{-\infty}^0 G_i(V)dV,$$

$$J_e^G = -e\int_0^{+\infty} VG_e(V)dV, \qquad J_i^G = e\int_{-\infty}^0 VG_i(V)dV,$$

$$W_e^G = m_e\int_0^{+\infty} \frac{V^2}{2}G_e(V)dV, \qquad W_i^G = m_i\int_{-\infty}^0 \frac{V^2}{2}G_i(V)dV.$$

For $\alpha \in \{e, i\}$, the thermal emission velocities are defined by

$$m_\alpha\frac{(V_\alpha^G)^2}{2}N_\alpha^G = W_\alpha^G$$

and we define the "electric" velocities by

$$m_\alpha\frac{(V_\alpha^L)^2}{2} = e\Phi_L.$$

As in [10, 11], we make the physical assumptions on which the Child-Langmuir asymptotics relies:

$$V_e^G \ll V_e^L \qquad V_i^G \ll V_i^L \tag{2.10}$$

$$J_e^G \text{ and } J_i^G \text{ are finite.} \tag{2.11}$$

We will now scale the system (2.1)-(2.9) in order to express the assumptions (2.10), (2.11) in a mathematical framework. For a justification of this scaling, we refer to [10, 11].

Here, the only point we have to take care of, is the fact that we are dealing with two species of particles. As we neglect all the collisions, the particles only interact through Poisson's equation (2.3). Hence, we can normalize the velocities and the currents using two different scaling factors, in order to take into account the differences of inertia of the two species. However, the densities are normalized by the same factor. We define the scaling units

$$\overline{N} = \frac{\varepsilon_0 \Phi_L}{eL^2}, \qquad \overline{J}_\alpha = e\overline{N}V_\alpha^L, \qquad \overline{F}_\alpha = \frac{\overline{N}}{V_\alpha^L},$$

and the scaled variables and unknowns for both particles

$$X = Lx, \qquad \Phi(X) = \Phi_L \varphi^\varepsilon(x), \qquad N_\alpha(X) = \overline{N}n^e ps_\alpha(x).$$

For the electrons we set

$$V = V_e^L v, \qquad J_e(X) = -\overline{J}_e\ j_e^\varepsilon(x), \qquad F_e(X,V) = \overline{F}_e\ f_e^\varepsilon(x,v),$$

and for the ions we take

$$V = V_i^L v, \qquad J_i(X) = \overline{J}_i\ j_i^\varepsilon(x), \qquad F_i(X,V) = \overline{F}_i\ f_i^\varepsilon(x,v).$$

To scale the boundary data, we have to introduce some small parameters. Using (2.10), we define

$$\varepsilon_e = \frac{V_e^G}{V_e^L} \ll 1, \qquad \varepsilon_i = \frac{V_i^G}{V_i^L} \ll 1,$$

and we introduce two dimensionless injection profiles $g_e(v)$ and $g_i(v)$,

$$g_e^\varepsilon(v) = \frac{1}{\varepsilon_e^2} g_e\left(\frac{v}{\varepsilon_e}\right), \quad g_i^\varepsilon(v) = \frac{1}{\varepsilon_i^2} g_i\left(\frac{v}{\varepsilon_i}\right),$$

so that

$$G_e(V) = \overline{F}_e\ g_e^\varepsilon\left(\frac{V}{V_e^L}\right), \quad G_i(V) = \overline{F}_i\ g_i^\varepsilon\left(\frac{V}{V_i^L}\right).$$

As in [10, 11], we can verify, *a posteriori*, that if the functions g_e and g_i are of order $\mathcal{O}(1)$, then this scaling is consistent with the assumptions (2.10), (2.11): the particles are injected with very small average velocities but very large densities, so that their currents remain finite. We set

$$j_e^g = \int_0^{+\infty} v g_e^\varepsilon(v)dv = \int_0^{+\infty} v g_e(v)dv \quad \text{and} \quad j_i^g = \int_{-\infty}^0 v g_i^\varepsilon(v)dv = \int_{-\infty}^0 v g_i(v)dv.$$

Notice that two small parameters ε_e and ε_i have been introduced. In the following we shall not need to compare them and we will let them tend to zero uniformly:

$$\varepsilon = \max(\varepsilon_e, \varepsilon_i) \longrightarrow 0.$$

In fact, from a physical point of view, these two small parameters are comparable in many cases, since the two species of particles have kinetic energies of the same order.

With the new variables and unknowns, and if $\Omega = [0,1] \times \mathbb{R}$, the dimensionless system writes

$$\begin{cases} v\dfrac{\partial f_e^\varepsilon}{\partial x} + \dfrac{1}{2}\dfrac{d\varphi^\varepsilon}{dx}\dfrac{\partial f_e^\varepsilon}{\partial v} = 0, & (x,v) \in \Omega, \quad (2.12) \\ v\dfrac{\partial f_i^\varepsilon}{\partial x} - \dfrac{1}{2}\dfrac{d\varphi^\varepsilon}{dx}\dfrac{\partial f_i^\varepsilon}{\partial v} = 0, & (x,v) \in \Omega, \quad (2.13) \\ \dfrac{d^2\varphi^\varepsilon}{dx^2} = n_e^\varepsilon - n_i^\varepsilon, & x \in [0,1], \quad (2.14) \end{cases}$$

$$n_e^\varepsilon(x) = \int_{\mathbf{R}} f_e^\varepsilon(x,v)dv, \qquad n_i^\varepsilon(x) = \int_{\mathbf{R}} f_i^\varepsilon(x,v)dv, \tag{2.15}$$

$$\begin{cases} f_e^\varepsilon(0,v) = g_e^\varepsilon(v) = \dfrac{1}{\varepsilon_e^2} g_e\left(\dfrac{v}{\varepsilon_e}\right), & v > 0, \quad (2.16) \\ f_e^\varepsilon(1,v) = 0, & v < 0, \quad (2.17) \\ f_i^\varepsilon(0,v) = 0, & v > 0, \quad (2.18) \\ f_i^\varepsilon(1,v) = g_i^\varepsilon(v) = \dfrac{1}{\varepsilon_i^2} g_i\left(\dfrac{v}{\varepsilon_i}\right) & v < 0, \quad (2.19) \\ \varphi^\varepsilon(0) = 0, \qquad \varphi^\varepsilon(1) = 1. & \quad (2.20) \end{cases}$$

We define the macroscopic quantities

$$j_e^\varepsilon(x) = \int_{\mathbf{R}} v f_e^\varepsilon(x,v)\ dv, \qquad j_i^\varepsilon(x) = \int_{\mathbf{R}} v f_i^\varepsilon(x,v)\ dv, \tag{2.21}$$

$$k_e^\varepsilon(x) = \int_{\mathbf{R}} v^2 f_e^\varepsilon(x,v)\ dv, \qquad k_i^\varepsilon(x) = \int_{\mathbf{R}} v^2 f_i^\varepsilon(x,v)\ dv, \tag{2.22}$$

$$h^\varepsilon = k_e^\varepsilon + k_i^\varepsilon - \frac{1}{4}\left(\frac{d\varphi^\varepsilon}{dx}\right)^2. \tag{2.23}$$

Since we do not know *a priori* the shape of the potential φ^ε, we cannot provide any explicit formula of the distribution function as in [10] cannot be adapted here.

Therefore, we follow the method of Degond, Jaffard, Poupaud and Raviart [9] and we estimate the distribution functions using super-solutions to the Vlasov equation.

For a fixed $\varepsilon > 0$, the following theorem shows that Problem (2.12)-(2.20) is well-posed. Its proofs is a straightforward adaptation of the results of Poupaud [17]. We assume that

$$0 \le g_\alpha(v) \le M(1+v^2)^{-\gamma}, \quad \alpha \in \{e,i\}, \quad \gamma > 1/2. \tag{2.24}$$

Since in realistic conditions the injection profiles vanish exponentially at infinity, this hypothesis is physically reasonable. In fact, in this paper we shall only need the three first moments of the distribution functions (of order 0, 1 and 2), so γ is further required to satisfy

$$\gamma > 3/2. \tag{2.25}$$

Theorem 2.1 *Let $\varepsilon > 0$ be fixed. If Condition* (2.24) *is satisfied, then the Vlasov-Poisson system with two species of particles* (2.12)-(2.20) *has at least one weak solution $(f_e^\varepsilon, f_i^\varepsilon, \varphi^\varepsilon)$, that verifies*

$$\begin{array}{l} f_\alpha^\varepsilon \in L^1(\Omega) \cap L^\infty(\Omega), \quad \textit{for } \alpha \in \{e,i\}, \\ \varphi^\varepsilon \in W^{2,\infty}([0,1]), \end{array}$$

Moreover, if G is a C^1 function such that

$$0 \le g_\alpha(v) \le G(v^2), \quad \textit{for } \alpha \in \{e,i\},$$

then these distribution functions satisfy

$$0 \le f_e^\varepsilon(x,v) \le \frac{1}{\varepsilon_e^2} G\left(\frac{v^2 - \varphi^\varepsilon(x)}{\varepsilon_e^2}\right),$$

$$0 \le f_i^\varepsilon(x,v) \le \frac{1}{\varepsilon_i^2} G\left(\frac{v^2 - 1 + \varphi^\varepsilon(x)}{\varepsilon_i^2}\right).$$

Finally we recall the invariance of some macroscopic quantities defined above. This result is directly obtained from the Vlasov-Poisson system, after some integrations of the Vlasov equation and the inequalities below are consequences of the boundary conditions (2.16) and (2.19). Of course, f_e^ε and f_i^ε are only $L^\infty \cap L^1$, but their traces are well defined thanks to the weak formulations of both Vlasov equations, which also allow to compare the currents inside the diode to the injected currents at the boundaries (see [9] and [17] for details).

Lemma 2.2 *$j_e^\varepsilon, j_i^\varepsilon$ and h^ε are independent of x. Moreover we have*

$$0 \le j_e^\varepsilon \le j_e^g \qquad \textit{and} \qquad j_i^g \le j_i^\varepsilon \le 0. \tag{2.26}$$

2.2. Statement of the main results

2.2.1. Convergence as $\varepsilon \to 0$

The following theorem gives the limiting behaviour of solutions to (2.12)-(2.20), as ε tends to 0.

Theorem 2.3 *Let $(f_e^\varepsilon, f_i^\varepsilon, \varphi^\varepsilon)$ be a solution of* (2.12)-(2.20) *given by Theorem* 2.1, *and let $(n_e^\varepsilon, n_i^\varepsilon, j_e^\varepsilon, j_i^\varepsilon)$ be the different moments defined previously. Then, for $\alpha \in \{e,i\}$, we have, as $\varepsilon \to 0$ and after extraction of subsequences,*

$$
\begin{gathered}
f_\alpha^\varepsilon \to f_\alpha \quad \text{in } \mathcal{M}_b(\Omega) \text{ weak } \star, \\
n_\alpha^\varepsilon \to n_\alpha \quad \text{in } L^1([0,1]) \text{ strong}, \\
j_\alpha^\varepsilon \to j_\alpha, \\
\varphi^\varepsilon \to \varphi \quad \text{in } W^{2,1}([0,1]) \text{ strong},
\end{gathered}
$$

where $\mathcal{M}_b(\Omega)$ denotes the space of bounded measures on $\Omega = [0,1] \times \mathbb{R}$. The limit densities, currents and distribution functions verify

$$
0 < j_e \le j_e^g, \quad j_i^g \le j_i < 0, \tag{2.27}
$$

$$
n_e(x) = \frac{j_e}{\sqrt{\varphi(x)}}, \quad n_i(x) = \frac{-j_i}{\sqrt{1-\varphi(x)}}, \tag{2.28}
$$

$$
f_e(x,v) = n_e(x)\ \delta(v - \sqrt{\varphi(x)}), \quad f_i(x,v) = n_i(x)\ \delta(v - \sqrt{1-\varphi(x)}). \tag{2.29}
$$

Moreover, f_e, f_i are weak solutions of the Vlasov equations

$$
\begin{cases}
v\dfrac{\partial f_e}{\partial x} + \dfrac{1}{2}\dfrac{d\varphi}{dx}\dfrac{\partial f_e}{\partial v} = 0, \\[2ex]
v\dfrac{\partial f_i}{\partial x} - \dfrac{1}{2}\dfrac{d\varphi}{dx}\dfrac{\partial f_i}{\partial v} = 0
\end{cases} \tag{2.30}
$$

and the potential φ is solution of

$$
\begin{cases}
\dfrac{d^2\varphi}{dx^2} = n_e(x) - n_i(x) = \dfrac{j_e}{\sqrt{\varphi(x)}} + \dfrac{j_i}{\sqrt{1-\varphi(x)}}, \quad x \in [0,1] & (2.31) \\[2ex]
\varphi(0) = 0, \quad \varphi(1) = 1. & (2.32)
\end{cases}
$$

Formulae (2.29) show that the limit distribution functions are those of monokinetic beams, emitted at rest from the electrodes.

Moreover, the two currents j_e and j_i have to verify the additional constraint

$$
\begin{cases}
0 < j_e \le j_e^g, \quad j_i^g \le j_i < 0, \\
\varphi'(0) > 0 \Longrightarrow j_e = j_e^g \\
\varphi'(1) > 0 \Longrightarrow j_i = j_i^g.
\end{cases} \tag{2.33}
$$

This property will be shown in Lemma 4.1. In the sequel, a triplet $(j_e, -j_i, \varphi)$ will be called solution of the limit model if it verifies (2.31), (2.32) and (2.33).

2.2.2. Analysis of the limiting model

We first recall the classical result for one species of particle [10]. For instance, assume that no ion is injected at the anode: $g_i^\varepsilon \equiv 0$, so that $j_i^g = 0$. Then we have

$$j_e = \lim_{\varepsilon \to 0} j_e^\varepsilon = \inf(j_e^g, j_1^{\mathrm{CL}}),$$

where $j_1^{\mathrm{CL}} = 4/9$ is the dimensionless one-species Child-Langmuir limiting current. If $j_e^g < j_1^{\mathrm{CL}}$, then the emission of particles is said to be *source limited*. Otherwise, if $j_e^g \geq j_1^{\mathrm{CL}}$, the emission is *space-charge limited*. In this case, we also say that the diode operates under the Child-Langmuir regime, and the important property is that the electric field $-\varphi'$ vanishes at the emitting electrode.

To present the results concerning the limit model in our case, we introduce the dimensionless bipolar Child-Langmuir current

$$j_2^{\mathrm{CL}} = \frac{1}{2}\left(\int_0^1 \frac{du}{\sqrt{\sqrt{u}+\sqrt{1-u}-1}}\right)^2 \simeq 1.865\ j_1^{\mathrm{CL}}.$$

We divide the quarter-plane $(j_e^g, -j_i^g) \in \mathbb{R}_+^*$, represented on Figure 1, into four domains, denoted by (i), (ii), (iii), (iv). The dividing lines are two curves $(\mathcal{C}_e)$, $(\mathcal{C}_i)$, which are defined in 4.1., and the two half-lines

$$\{j_e^g = j_2^{\mathrm{CL}},\ -j_i^g \geq j_2^{\mathrm{CL}}\} \text{ and } \{-j_i^g = j_2^{\mathrm{CL}},\ j_e^g \geq j_2^{\mathrm{CL}}\}.$$

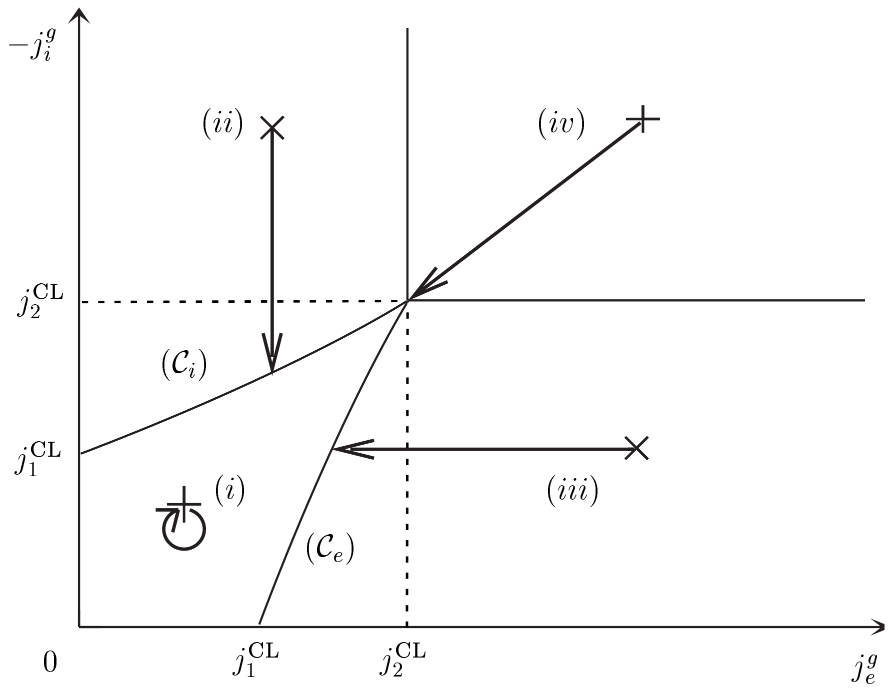

Figure 1 : *Domains of the different regimes*

We first state a theorem, that describes the situation when the injected currents are not two large. Indeed if we have

$$j_e^g < 9 j_2^{\mathrm{CL}} \quad or \quad - j_i^g < 9 j_2^{\mathrm{CL}}, \tag{$\mathcal{H}$}$$

then the limit model possesses a unique solution characterized by the following theorem:

Theorem 2.4 *Let $j_e^g > 0$, $j_i^g < 0$ be such that $(\mathcal{H})$ is verified. Then, there exists a unique solution (j_e, j_i, φ) to the limiting model* (2.31), (2.32), (2.33). *Moreover the potential φ is increasing and the four cases can occur, according to the position of the pair $(j_e^g, -j_i^g)$ in $(\mathbb{R}_+^*)^2$:*
- *if $(j_e^g, -j_i^g) \in (i)$, both emissions are source limited and $(j_e, j_i) = (j_e^g, j_i^g)$;*
- *if $(j_e^g, -j_i^g) \in (ii)$, the emission of electrons is source limited, $j_e = j_e^g$, and the emission of ions is space-charge limited, such that $(j_e^g, -j_i) \in (\mathcal{C}_i)$;*
- *if $(j_e^g, -j_i^g) \in (iii)$, the emission of ions is source limited, $j_i = j_i^g$, and the emission of electrons is space-charge limited, such that $(j_e, -j_i^g) \in (\mathcal{C}_e)$;*
- *if $(j_e^g, -j_i^g) \in (iv)$, the emissions of electrons and ions are space-charge limited; $j_e = -j_i = j_2^{\mathrm{CL}}$ and $\varphi'(0) = \varphi'(1) = 0$.*

Remark that in this case $(\mathcal{H})$, since the solution is unique, all the sequence $(\varphi^\varepsilon, f_e^\varepsilon, f_i^\varepsilon)$ converges to its limit and not only a subsequence as stated in Theorem 2.3. Theorem 2.4 is summed up on Figure 1, where the arrows represent the four different cases. Each arrow joins, at its base, the pair of data $(j_e^g, -j_i^g)$ and, at its end, the pair $(j_e, -j_i)$ of effective currents flowing inside the diode.

Case (iv) is the most interesting physical case. This case, where both currents are space-charge limited, is called the Child-Langmuir regime. On Figure 2, we plotted the profile of the potential in this regime, denoted φ_{CL}.

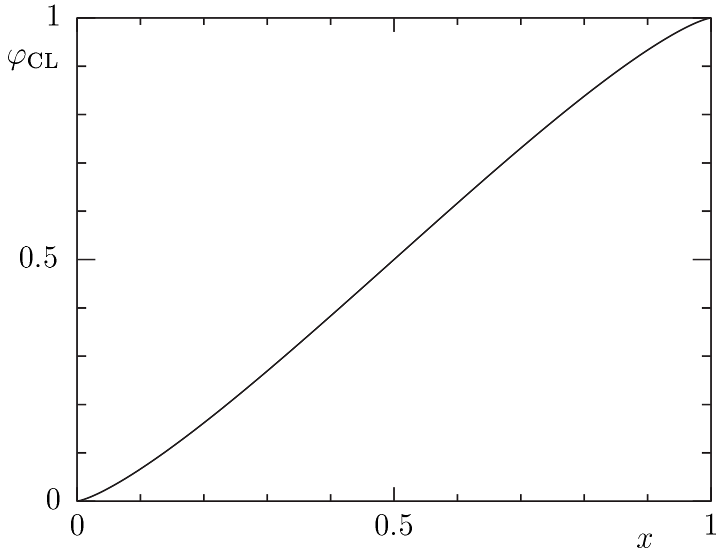

Figure 2 : *Potential in the Child-Langmuir regime*

We now describe the results obtained in the case of large currents, when the assumption $(\mathcal{H})$ is not verified. The limiting solution is not unique any more, but all the solutions can be constructed on the basis of the monotone solution $\varphi_{\rm CL}$.

Theorem 2.5 *Let j_e^g and $-j_i^g$ be two arbitrary positive constants verifying*

$$\min(j_e^g, -j_i^g) \geq 9\ j_2^{\rm CL}. \qquad (\overline{\mathcal{H}})$$

Then there exists an integer p that verifies

$$(2p+1)^2 j_2^{\rm CL} \leq \min(j_e^g, -j_i^g)$$

and such that the solution of the limit model (2.31), (2.32), (2.33) *is described as follows:*

• *If $p = 0$ then $\varphi \equiv \varphi_{\rm CL}$ and $j_e = -j_i = j_2^{\rm CL}$: it is exactly the solution given in Theorem* 2.4*, case (iv).*

• *If $p \geq 1$ then the currents are $j_e = -j_i = (2p+1)^2 j_2^{\rm CL}$. Let $r_k = k/(2p+1)$ for $0 \leq k \leq 2p+1$; the potential is defined by*

$$\begin{cases} \textit{for } x \in [r_{2k}, r_{2k+1}], & 0 \leq k \leq p, \quad \varphi(x) = \varphi_{\rm CL}\left((2p+1)(x - r_{2k})\right) \\ \textit{for } x \in [r_{2k+1}, r_{2k+2}], & 0 \leq k \leq p-1, \quad \varphi(x) = \varphi_{\rm CL}\left((2p+1)(r_{2k+2} - x)\right). \end{cases}$$

In all these cases ($p \geq 0$), the emissions of electrons and ions are space-charge limited.

We represented on Figure 3 the profile of the potential in the case $p = 2$.

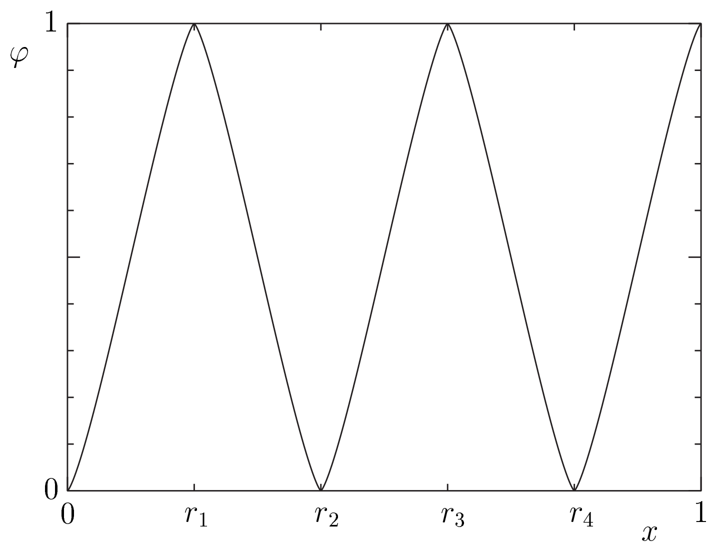

Figure 3 : *Potential in the case $p = 2$*

It appears a difference between these "parasitic" solutions, for $p \geq 1$, and the ones obtained in the case of electrons flowing in a background of positive ions [2]. Indeed, here the multiple solutions are completely characterized as soon as the integer p is given: p is the only degree of freedom. Moreover, an important feature is that in our case, one can see on the limit distribution functions (2.29) that the electrons (resp. the ions) that flow inside the diode can only have negative velocities (resp. positive). This does not allow the presence of "trapped particles" inside the diode as it was the case in [2], for a large injection electron current and a large doping density.

3. Convergence to the reduced model

This section is devoted to the proof of Theorem 2.3. In this, we mainly follow the outlines of [9] or [2].

In the following subsection, we obtain some uniform a priori estimates on the solution of the singular perturbation problem (2.12)-(2.20), basically by using the super-solution property stated in Theorem 2.1. These estimates will lead to a first convergence result, as ε goes to zero, stated in Proposition 3.4. Unfortunately, as in [9], [2], this convergence is too weak to precise completely the limit model, and the second subsection is devoted to the proof of a stronger convergence.

Here, unlike in these papers, we will not make the assumption of compactness for the supports of the injection profiles g_e and g_i. We only assume that (2.24) and (2.25) hold. In this, we justify the fact that, as written in [8], "it seems to be only technical matter to extend the proof [from compactly supported data] to boundary data with suitable decay at the infinity".

3.1. A priori estimates and first convergence result

Lemma 3.1 *If g_e and g_i verify* (2.24) *and if $(f_e^\varepsilon, f_i^\varepsilon, \varphi^\varepsilon)$ is a solution given by Theorem* 2.1 *then*

$$supp(f_e^\varepsilon) \subset \Big\{(x,v): \ 0 \le x \le 1 \quad and \quad 0 \le v^2 - \varphi^\varepsilon(x)\Big\}, \tag{3.1}$$

$$supp(f_i^\varepsilon) \subset \Big\{(x,v): \ 0 \le x \le 1 \quad and \quad 0 \le v^2 - 1 + \varphi^\varepsilon(x)\Big\}, \tag{3.2}$$

$$0 \le f_e^\varepsilon(x,v) \le \frac{M}{\varepsilon_e^2}\left(1 + \frac{|v^2 - \varphi^\varepsilon(x)|}{\varepsilon_e^2}\right)^{-\gamma}, \tag{3.3}$$

$$0 \le f_i^\varepsilon(x,v) \le \frac{M}{\varepsilon_i^2}\left(1 + \frac{|v^2 - 1 + \varphi^\varepsilon(x)|}{\varepsilon_i^2}\right)^{-\gamma}. \tag{3.4}$$

Proof. Since the problem (2.12)-(2.20) is symmetric with respect to $(f_e^\varepsilon, \varphi^\varepsilon)$ and $(f_i^\varepsilon, 1 - \varphi^\varepsilon)$, we will only prove here the formulae corresponding to the electrons, (3.1) and (3.3).

Let $V_0 > 0$ and G^{V_0} be a C^1 function that satisfies

$$\begin{cases} G^{V_0}(w) = 0 & \text{for } w \leq -V_0 \\ G^{V_0}(w) = M(1+w)^{-\gamma} & \text{for } w \geq 0. \end{cases}$$

Apply Theorem 2.1 and use (2.24), then let $V_0 \to 0$ to deduce the result. $\square$

As a consequence of the above results, we obtain the following uniform estimates; here and in all the sequel, C denotes a generic constant independent of ε.

Lemma 3.2 *If* $(f_e^\varepsilon, f_i^\varepsilon, \varphi^\varepsilon)$ *is a solution of* (2.12)-(2.20) *given by Theorem* 2.1*, and if* (2.25) *is verified, then the following estimates hold, for* $0 \leq x \leq 1$*:*

(i)

$$\text{if} \quad \varphi^\varepsilon \leq 0 \quad \text{then} \quad n_e^\varepsilon(x) \leq \frac{C}{\varepsilon_e}\left(\frac{\varepsilon_e}{\sqrt{-\varphi^\varepsilon + \varepsilon_e^2}}\right)^{2\gamma-1}, \tag{3.5}$$

$$\text{if} \quad \varphi^\varepsilon \geq 0 \quad \text{then} \quad n_e^\varepsilon(x) \leq \frac{C}{\sqrt{\varphi^\varepsilon + \varepsilon_e^2}}, \tag{3.6}$$

$$\text{if} \quad \varphi^\varepsilon \geq 1 \quad \text{then} \quad n_i^\varepsilon(x) \leq \frac{C}{\varepsilon_i}\left(\frac{\varepsilon_i}{\sqrt{\varphi^\varepsilon - 1 + \varepsilon_i^2}}\right)^{2\gamma-1}, \tag{3.7}$$

$$\text{if} \quad \varphi^\varepsilon \leq 1 \quad \text{then} \quad n_i^\varepsilon(x) \leq \frac{C}{\sqrt{1 - \varphi^\varepsilon + \varepsilon_i^2}}, \tag{3.8}$$

(ii)

$$|k_e^\varepsilon - n_e^\varepsilon \varphi^\varepsilon| \leq C\varepsilon_e, \tag{3.9}$$

$$|k_i^\varepsilon - n_i^\varepsilon(1 - \varphi^\varepsilon)| \leq C\varepsilon_i, \tag{3.10}$$

(iii)

$$0 \leq k_e^\varepsilon \leq C(\varepsilon_e + \sqrt{|\varphi^\varepsilon|}), \tag{3.11}$$

$$0 \leq k_i^\varepsilon \leq C(\varepsilon_i + \sqrt{|1 - \varphi^\varepsilon|}). \tag{3.12}$$

Proof. For $k = 0, 1$ or 2, denote $I_\gamma^{(k)} = \displaystyle\int_0^{+\infty} w^k (1 + w^2)^{-\gamma}\, dw$; this integral is well defined thanks to (2.25).
(i) Because of the symmetry of the problem, we will again only prove the results concerning the electrons: (3.5), (3.6), (3.9) and (3.11). Moreover, for the sake of simplicity of the expressions, and since we will have no ε_i here, we will replace ε_e by ε in the proof of this lemma.

From (3.1) and (3.3) we get

$$\begin{aligned} n_e^\varepsilon &= \int_{v^2 - \varphi^\varepsilon \geq 0} f_e^\varepsilon(x,v)\,dv \\ &\leq \frac{2M}{\varepsilon^2} \int_{\substack{v^2 - \varphi^\varepsilon \geq 0 \\ v \geq 0}} \left(1 + \frac{v^2 - \varphi^\varepsilon(x)}{\varepsilon^2}\right)^{-\gamma} dv \; = 2M.\mathcal{I}. \end{aligned}$$

To estimate the integral $\mathcal{I}$, we distinguish the two cases, whether φ^ε is positive or negative.

• Let $\varphi^\varepsilon \leq 0$. The integral $\mathcal{I}$ is in fact calculated on $[0,+\infty[$ and, after the change of variable

$$w = \frac{v}{\sqrt{\varepsilon^2 - \varphi^\varepsilon}},$$

it writes exactly

$$\mathcal{I} = \frac{I_\gamma^{(0)}}{\varepsilon}\left(\frac{\varepsilon}{\sqrt{-\varphi^\varepsilon + \varepsilon^2}}\right)^{2\gamma-1};$$

(3.5) is proved.

• Let $\varphi^\varepsilon \geq 0$. With the change of variable

$$w = \frac{\sqrt{v^2 - \varphi^\varepsilon}}{\varepsilon},$$

we have

$$\mathcal{I} = \frac{1}{\varepsilon^2}\int_{\sqrt{\varphi^\varepsilon}}^{+\infty}\left(1 + \frac{v^2 - \varphi^\varepsilon}{\varepsilon^2}\right)^{-\gamma} dv = \int_0^{+\infty}(1+w^2)^{-\gamma}\frac{w}{\sqrt{\varepsilon^2 w^2 + \varphi^\varepsilon}}dw.$$

For $\varphi^\varepsilon \leq \varepsilon^2$, this integral can be estimated by

$$\mathcal{I} \leq \frac{I_\gamma^{(0)}}{\varepsilon} \leq \frac{I_\gamma^{(0)}\sqrt{2}}{\sqrt{\varepsilon^2 + \varphi^\varepsilon}}.$$

For $\varphi^\varepsilon \geq \varepsilon^2$, we write

$$\mathcal{I} \leq \int_0^{+\infty}(1+w^2)^{-\gamma}\frac{w}{\sqrt{\varphi^\varepsilon}}dw = \frac{I_\gamma^{(1)}}{\sqrt{\varphi^\varepsilon}} \leq \frac{I_\gamma^{(1)}\sqrt{2}}{\sqrt{\varepsilon^2 + \varphi^\varepsilon}}.$$

The estimate (3.6) follows.

(ii) As for the proof of *(i)*, we use (3.1) and (3.3):

$$\begin{aligned} 0 \leq k_e^\varepsilon(x) - n_e^\varepsilon(x)\varphi^\varepsilon(x) &= \int_{v^2 - \varphi^\varepsilon \geq 0}(v^2 - \varphi^\varepsilon(x))f_e^\varepsilon(x,v)dv \\ &\leq 2M \int_{\substack{v^2 - \varphi^\varepsilon \geq 0 \\ v \geq 0}} \frac{\frac{v^2 - \varphi^\varepsilon(x)}{\varepsilon^2}}{\left(1 + \frac{v^2 - \varphi^\varepsilon(x)}{\varepsilon^2}\right)^\gamma}\, dv \ = 2M.\mathcal{I}'. \end{aligned}$$

We distinguish again two cases.

• Let $\varphi^\varepsilon \geq 0$. With the same change of variable as above $\varepsilon^2 w^2 = v^2 - \varphi^\varepsilon$,

$$\mathcal{I}' = \varepsilon^2 \int_0^{+\infty} \frac{w^3\, dw}{(1+w^2)^\gamma\, \sqrt{\varepsilon^2 w^2 + \varphi^\varepsilon}} \leq \varepsilon \int_0^{+\infty} \frac{w^2}{(1+w^2)^\gamma}\, dw = \varepsilon I_\gamma^{(2)};$$

thus

$$|k_e^\varepsilon - n_e^\varepsilon \varphi^\varepsilon| \leq 2M I_\gamma^{(2)} \varepsilon.$$

• Otherwise $\varphi^\varepsilon \leq 0$. In fact we write

$$|k_e^\varepsilon - n_e^\varepsilon \varphi^\varepsilon| = |k_e^\varepsilon| + |-n_e^\varepsilon \varphi^\varepsilon|.$$

For the second term, we multiply (3.5) by $(-\varphi^\varepsilon)$; since $\gamma > 3/2$ and $-\varphi^\varepsilon \geq 0$, we have

$$0 \leq (-\varphi^\varepsilon n_e^\varepsilon) \leq \frac{C(-\varphi^\varepsilon)}{\varepsilon} \left(\frac{\varepsilon}{\sqrt{-\varphi^\varepsilon + \varepsilon^2}} \right)^2 \leq C\varepsilon.$$

The first term is estimated as follows

$$\begin{aligned} 0 \leq k_e^\varepsilon \quad &\leq 2M \int_0^{+\infty} \frac{v^2}{\varepsilon^2} \left(1 - \frac{\varphi^\varepsilon}{\varepsilon^2} + \frac{v^2}{\varepsilon^2} \right)^{-\gamma} dv \\ &\leq 2M \int_0^{+\infty} \frac{v^2}{\varepsilon^2} \left(1 + \frac{v^2}{\varepsilon^2} \right)^{-\gamma} dv = 2M \ I_\gamma^{(2)} \ \varepsilon. \end{aligned}$$

This concludes the proof of (ii).

(iii) is a direct consequence of (i) and (ii). □

We can now prove some other estimates, based on Lemmas 2.2, 3.2 and on Poisson's equation (2.14).

Lemma 3.3 *The following estimates hold, for ε small enough:*

(i)
$$-C\varepsilon_e^\beta \leq \varphi^\varepsilon(x) \leq 1 + C\varepsilon_i^\beta, \quad \textit{where } \beta = 2 - \frac{3}{\gamma + 1/2}, \tag{3.13}$$

(ii)
$$\left| \frac{d\varphi^\varepsilon}{dx} \right| \leq C, \tag{3.14}$$

(iii)
$$\|n_e^\varepsilon\|_{L^1(0,1)} + \|n_i^\varepsilon\|_{L^1(0,1)} \leq C. \tag{3.15}$$

Proof.
(i) Remark first that if $\gamma \to +\infty$, i.e. the injection profiles vanish very quickly at $+\infty$, then $\beta \to 2$. This was the exponent obtained in [9], [2], in the estimate corresponding to (3.13) when the injection profiles were supposed to be compactly supported.

Thanks to the symmetry of the problem, we shall only prove the under- bound of φ^ε. Consider a^ε such that $\varphi^\varepsilon(a^\varepsilon) = \min\limits_{[0,1]}(\varphi^\varepsilon)$. If $\varphi^\varepsilon(a^\varepsilon) \geq 0$ then the estimate follows directly. In the sequel of this proof, one can thus assume $\varphi^\varepsilon(a^\varepsilon) < 0$. Setting

$$\beta = 2 - \frac{3}{\gamma + 1/2} \text{ and } K^\varepsilon = \frac{\varphi^\varepsilon(a^\varepsilon)}{-2\varepsilon^\beta}, \tag{3.16}$$

we have

$$\varphi^\varepsilon(a^\varepsilon) = -2K^\varepsilon \varepsilon^\beta. \tag{3.17}$$

Therefore, since φ^ε is continuous, we can define the maximal open interval $]b^\varepsilon, c^\varepsilon[\subset]0,1[$ containing a^ε and verifying the property

$$\forall x \in]b^\varepsilon, c^\varepsilon[\qquad \varphi^\varepsilon(x) < -K^\varepsilon \varepsilon^\beta. \tag{3.18}$$

With the boundary conditions on φ^ε, (2.20), we can already say that

$$0 < b^\varepsilon < a^\varepsilon < c^\varepsilon < 1$$

which implies

$$\varphi^\varepsilon(b^\varepsilon) = \varphi^\varepsilon(c^\varepsilon) = -K^\varepsilon \varepsilon^\beta. \tag{3.19}$$

Moreover, with (3.5) and (3.18), it comes

$$\begin{aligned} \forall x \in [b^\varepsilon, c^\varepsilon], \qquad n_e^\varepsilon \; &\leq \frac{C}{\varepsilon_e} \left(\frac{\varepsilon_e}{\sqrt{K^\varepsilon \varepsilon^\beta + \varepsilon_e^2}} \right)^{2\gamma-1} \\ &\leq C \; (K^\varepsilon)^{1/2-\gamma} \; \varepsilon_e^{2\gamma-2} \; \varepsilon^{\beta(1/2-\gamma)} \\ &\leq C \; (K^\varepsilon)^{1/2-\gamma} \; \varepsilon^\beta = n_0. \end{aligned}$$

Remark that the value of β (3.16) has been adjusted such that $\beta = \beta(1/2-\gamma)+2\gamma-2$. Next we report this estimate into Poisson's equation (2.14)

$$\forall x \in [b^\varepsilon, c^\varepsilon], \qquad \frac{d^2\varphi^\varepsilon}{dx^2} = n_e^\varepsilon - n_i^\varepsilon \leq n_0,$$

then integrate this expression twice between a^ε and x; with $d\varphi^\varepsilon/dx(a^\varepsilon) = 0$ and (3.17) we get

$$\begin{aligned} \forall x \in [b^\varepsilon, c^\varepsilon], \quad \varphi^\varepsilon(x) \; &\leq \frac{(x-a^\varepsilon)^2}{2} n_0 - 2K^\varepsilon \varepsilon^\beta \\ &\leq \frac{n_0}{2} - 2K^\varepsilon \varepsilon^\beta = \left(\frac{C \; (K^\varepsilon)^{-1/2-\gamma}}{2} - 2 \right) K^\varepsilon \varepsilon^\beta. \end{aligned}$$

Write it at the point b^ε. (3.19) implies

$$K^\varepsilon \leq \left(\frac{C}{2} \right)^{\frac{1}{\gamma+1/2}}.$$

Hence, with the definition of a^ε and (3.17), we deduce that

$$\forall x \in [0,1], \qquad \varphi^\varepsilon(x) \geq -2 \left(\frac{C}{2} \right)^{\frac{1}{\gamma+1/2}} \varepsilon^\beta.$$

(ii) First, we deduce directly from (3.11), (3.12) and (3.13) that

$$\|k_e^\varepsilon\|_{L^\infty(0,1)} + \|k_i^\varepsilon\|_{L^\infty(0,1)} \leq C.$$

Thus, to obtain (3.14), it is sufficient to prove the L^∞ bound of

$$h^\varepsilon = k_e^\varepsilon + k_i^\varepsilon - \frac{1}{4}\left(\frac{d\varphi^\varepsilon}{dx}\right)^2.$$

Lemma 2.2 says that h^ε is independent of x. We conclude by saying that there exists a point of $[0,1]$ where $\dfrac{d\varphi^\varepsilon}{dx} = \varphi^\varepsilon(1) - \varphi^\varepsilon(0) = 1$.

(iii) This estimate is crucial to get the convergence towards a certain model. With (3.9), (3.10) and the L^∞ bounds of k_e^ε and k_i^ε, we obtain

$$\|n_e^\varepsilon \varphi^\varepsilon\|_{L^\infty} + \|n_i^\varepsilon(1-\varphi^\varepsilon)\|_{L^\infty} \leq C. \tag{3.20}$$

Now writing

$$0 \leq n_i^\varepsilon = (n_e^\varepsilon \varphi^\varepsilon + n_i^\varepsilon(1-\varphi^\varepsilon)) - (n_e^\varepsilon - n_i^\varepsilon)\varphi^\varepsilon,$$

we deduce

$$\begin{aligned}\int_0^1 n_i^\varepsilon(x)dx &= \int_0^1 (n_e^\varepsilon\varphi^\varepsilon + n_i^\varepsilon(1-\varphi^\varepsilon)) - \int_0^1 \frac{d^2\varphi^\varepsilon}{dx^2}\varphi^\varepsilon \\ &= \int_0^1 (n_e^\varepsilon\varphi^\varepsilon + n_i^\varepsilon(1-\varphi^\varepsilon)) + \int_0^1 (\frac{d\varphi^\varepsilon}{dx})^2 - \frac{d\varphi^\varepsilon}{dx}(1),\end{aligned}$$

after an integration by parts, since $\varphi(0) = 0$ and $\varphi(1) = 1$. Thus, from (3.20) and (3.14), we conclude that

$$\|n_i^\varepsilon\|_{L^1(0,1)} = \int_0^1 n_i^\varepsilon dx \leq C.$$

We do the same for n_e^ε. $\square$

Finally, these lemmas enable to derive a first convergence result:

Proposition 3.4 *There exist functions $\varphi \in W^{1,\infty}([0,1])$, $(k_e, k_i) \in (L^\infty(0,1))^2$, bounded measures $(f_e, f_i) \in (\mathcal{M}_b(\Omega))^2$, $(n_e, n_i) \in (\mathcal{M}_b([0,1]))^2$, and real numbers j_e, j_i, h such that, after extraction of subsequences, as $\varepsilon \to 0$ and for $\alpha \in \{e, i\}$,*

$$\begin{cases} \varphi^\varepsilon \to \varphi \quad \text{in the } C^0[0,1] \text{ strong topology,} \\ \dfrac{d\varphi^\varepsilon}{dx} \to \dfrac{d\varphi}{dx} \quad \text{in } L^p(0,1) \text{ strong for all } p \in [1,\infty[\text{ and in } L^\infty \text{ weak } \star, \\ f_\alpha^\varepsilon \to f_\alpha \quad \text{in } \mathcal{M}_b(\Omega) \text{ weak } \star, \\ n_\alpha^\varepsilon \to n_\alpha \quad \text{in } \mathcal{M}_b([0,1]) \text{ weak } \star, \\ k_\alpha^\varepsilon \to k_\alpha \quad \text{in } L^\infty(0,1) \text{ weak } \star, \\ j_\alpha^\varepsilon \to j_\alpha, \\ h^\varepsilon \to h. \end{cases}$$

Proof. It is sufficient to remark that the previous lemmas give the estimates

$$\|\varphi^\varepsilon\|_{W^{1,\infty}\cap W^{2,1}([0,1])} \le C,$$

$$\|f_e^\varepsilon\|_{L^1(\Omega)} + \|f_i^\varepsilon\|_{L^1(\Omega)} \le C,$$

$$\|n_e^\varepsilon\|_{L^1(0,1)} + \|n_i^\varepsilon\|_{L^1(0,1)} \le C,$$

$$\|k_e^\varepsilon\|_{L^\infty(0,1)} + \|k_i^\varepsilon\|_{L^\infty(0,1)} \le C,$$

$$|j_e^\varepsilon| + |j_i^\varepsilon| + |h^\varepsilon| \le C.$$

□

From this proposition we can deduce the

Lemma 3.5 *The limit* $(\varphi, f_\alpha, n_\alpha, j_\alpha, k_\alpha, h)_{\alpha\in\{e,i\}}$ *given by Proposition* 3.4 *verifies*

$$0 \le \varphi(x) \le 1, \text{ for } x \in [0,1], \tag{3.21}$$

$$\mathrm{Supp}(f_e) \subset \Big\{(x,v):\ 0 \le x \le 1 \quad and \quad v^2 = \varphi(x)\Big\}, \tag{3.22}$$

$$\mathrm{Supp}(f_i) \subset \Big\{(x,v):\ 0 \le x \le 1 \quad and \quad v^2 = 1-\varphi(x)\Big\}, \tag{3.23}$$

$$\begin{cases} \dfrac{d^2\varphi}{dx^2} = n_e - n_i \quad in\ \mathcal{M}_b([0,1]), \\ \varphi(0) = 0, \quad \varphi(1) = 1, \end{cases} \tag{3.24}$$

$$n_\alpha = \langle f_\alpha, 1\rangle_{\mathcal{M}_b^0,C}, \quad j_\alpha = \langle f_\alpha, v\rangle_{\mathcal{M}_b^0,C}, \quad k_\alpha = \langle f_\alpha, v^2\rangle_{\mathcal{M}_b^0,C}, \tag{3.25}$$

$$h = k_e + k_i - \frac{1}{4}\left(\frac{d\varphi}{dx}\right)^2. \tag{3.26}$$

In this lemma, we denote by $\langle .,.\rangle_{\mathcal{M}_b^0,C}$ the duality product between compactly supported bounded measures and continuous functions.

Proof. For (3.21), (3.24), (3.25) and (3.26), we proceed as in [9], by passing to the limit in the corresponding inequalities; these estimates derive respectively from {(3.13)} , {(2.14),(2.20)}, {(2.15),(2.21),(2.22)} and {(2.23)}.

We only detail the proof of (3.22). Let $\theta(x,v)$ be a compactly supported test function, such that

$$\mathrm{Supp}(\theta) \cap \Big\{(x,v):\ v^2 = \varphi(x)\Big\} = \emptyset.$$

By this property and the C^0 convergence of φ^ε, there exists $\eta > 0$ and $\varepsilon_1 > 0$ such that

$$\forall (x,v) \in \mathrm{Supp}(\theta),\ \forall \varepsilon < \varepsilon_1, \quad |v^2 - \varphi^\varepsilon(x)| \ge \eta.$$

Moreover, let us denote $L = ||\theta||_{L^1(0,1)\times\mathbf{R}}$. Hence, by (3.3), we have

$$\begin{aligned}
\langle f_e^\varepsilon, \theta\rangle_{\mathcal{M}_b^0,C} &= \int_0^1 \int_{-\infty}^{+\infty} f_e^\varepsilon(x,v)\ \theta(x,v)\ dxdv \\
&\leq \frac{M}{\varepsilon_e^2} \int_0^1 \int_{-\infty}^{+\infty} \left(1 + \frac{|v^2 - \varphi^\varepsilon(x)|}{\varepsilon_e^2}\right)^{-\gamma} \theta(x,v)\ dxdv \\
&\leq ML \frac{\varepsilon_e^{2\gamma-2}}{(\varepsilon_e^2+\eta)^\gamma} \leq ML \frac{\varepsilon_e^{2\gamma-2}}{\eta^\gamma}.
\end{aligned}$$

Therefore $\langle f_e^\varepsilon, \theta\rangle_{\mathcal{M}_b^0,C} \to 0$ as $\varepsilon_e \to 0$, which implies (3.22). The property (3.23) is obtained similarly, with (3.4).

□

3.2. Stronger convergence

The convergence result obtained above is not sufficient to give a precise description of the limit model, which is indeed the aim of this paper. For instance we cannot pass to the limit in the Vlasov equations (2.12), (2.13), since we did not prove the convergence of the limit potential in the $C^1([0,1])$ norm. At a point where the potential vanishes (resp. takes the value 1), Lemma 3.2 does not enable us to control the density of electrons n_e^ε (resp. of ions n_i^ε). It is important to show that the accumulation of particles near such a point does not create Dirac masses, i.e. that the limit densities n_e and n_i remain L^1 functions.

To prove that $\varphi^\varepsilon \to \varphi$ in the C^1 norm and that $n_\alpha^\varepsilon \to n_\alpha$ in L^1, we will consider different kinds of intervals. Indeed, our arguments will be different depending on whether we study neighbourhoods of the singular points, which are the points where φ takes the values 0 or 1, or whether the interval considered is far from these points. To decompose $[0,1]$ in such a way, the crucial step consists, as in [2], in proving that these singular points are isolated. Several lemmas will lead us to this conclusion.

For $\alpha \in \{e,i\}$, set

$$j_\alpha^{\varepsilon,+}(x) = \int_0^{+\infty} v f_\alpha^\varepsilon(x,v)dv, \quad j_\alpha^{\varepsilon,-}(x) = \int_{-\infty}^0 v f_\alpha^\varepsilon(x,v)dv = j_e^\varepsilon - j_e^{\varepsilon,+}(x).$$

Lemma 3.6 *Let* $\mathrm{I} = [a,b] \subset [0,1]$ *such that* $\varphi(x) > 0$ *on* I. *Then, for* ε *small enough, we have*

$$\frac{dj_e^{\varepsilon,+}}{dx} \equiv \frac{dj_e^{\varepsilon,-}}{dx} \equiv 0 \quad \text{on } \mathrm{I}$$

and

$$j_e^{\varepsilon,+} \to j_{e,\mathrm{I}}^+ \geq 0, \quad j_e^{\varepsilon,-} \to j_{e,\mathrm{I}}^- \leq 0 \text{ as } \varepsilon \to 0, \tag{3.27}$$

$j^+_{e,\mathrm{I}}$ and $j^-_{e,\mathrm{I}}$ being two constant real numbers. Moreover n^ε_e and k^ε_e converge uniformly to n_e and k_e on I *and, for $x \in \mathrm{I}$, we have*

$$j_e = j^+_{e,\mathrm{I}} + j^-_{e,\mathrm{I}}, \tag{3.28}$$

$$n_e(x) = \frac{j^+_{e,\mathrm{I}} - j^-_{e,\mathrm{I}}}{\sqrt{\varphi(x)}}, \tag{3.29}$$

$$k_e(x) = (j^+_{e,\mathrm{I}} - j^-_{e,\mathrm{I}})\sqrt{\varphi(x)}. \tag{3.30}$$

Proof. Let $\alpha = \min_{\mathrm{I}} \varphi > 0$ and $\alpha^\varepsilon = \min_{\mathrm{I}} \varphi^\varepsilon$; since by Proposition 3.4 φ^ε converges to φ in the $C^0([0,1])$ norm, we have $\alpha^\varepsilon > \alpha/2$ for ε small enough. Hence by (3.1) we have,

$$\text{for } x \in \mathrm{I} \text{ and } |v| < \sqrt{\frac{\alpha}{2}}, \quad f^\varepsilon_e(x,v) = 0.$$

Hence one deduces as in [9] Lemma 4.6, that $j^{\varepsilon,+}_e$ and $j^{\varepsilon,-}_e$ do not depend on x on the interval I. We do not develop this proof here (see [9] for details; this proof is based on a weak formulation of the Vlasov equation, with a suitable test function).

Let now θ be a continuous function, positive on $]a,b[$ and vanishing on $[0,a]\cup[b,1]$. We have

$$\langle f^\varepsilon_e, v^+\theta(x)\rangle_{\mathcal{M}^0_b,C} = j^{\varepsilon,+}_e \int_a^b \theta(x)dx.$$

Setting

$$j^+_{e,\mathrm{I}} = \frac{\langle f_e, v^+\theta(x)\rangle_{\mathcal{M}^0_b,C}}{\int_a^b \theta(x)dx},$$

by Proposition 3.4 $j^{\varepsilon,+}_e$ converges to $j^+_{e,\mathrm{I}}$ as $\varepsilon \to 0$ and this quantity does not depend on θ. Since $j^{\varepsilon,+}_e + j^{\varepsilon,-}_e = j^\varepsilon_e$, we also have $j^{\varepsilon,-}_e \to j^-_{e,\mathrm{I}} = j_e - j^+_{e,\mathrm{I}}$. (3.27) and (3.28) are proved.

To show (3.29) we apply Lemma 3.1. Recall that, on I, $\varphi^\varepsilon \geq \varepsilon^2$ for ε small enough. We have

$$\begin{aligned} 0 \leq j^{\varepsilon,+}_e - \sqrt{\varphi^\varepsilon}\int_0^{+\infty} f^\varepsilon_e dv &= \int_{\sqrt{\varphi^\varepsilon}}^{+\infty} (v - \sqrt{\varphi^\varepsilon}) f^\varepsilon_e(x,v)dv \\ &\leq \frac{M}{\varepsilon_e}\int_{\sqrt{\varphi^\varepsilon}}^{+\infty} \left(\frac{v-\sqrt{\varphi^\varepsilon}}{\varepsilon_e}\right)\left(1 + \frac{v^2 - \varphi^\varepsilon(x)}{\varepsilon_e^2}\right)^{-\gamma} dv. \end{aligned}$$

Let us perform the following variable substitution:

$$w = \frac{v - \sqrt{\varphi^\varepsilon}}{\varepsilon_e}.$$

We obtain directly

$$0 \leq j^{\varepsilon,+}_e - \sqrt{\varphi^\varepsilon}\int_0^{+\infty} f^\varepsilon_e dv \leq M\varepsilon_e^\gamma \int_0^{+\infty} \frac{w}{(\varepsilon_e + \varepsilon_e w^2 + 2\sqrt{\varphi^\varepsilon}w)^\gamma} dw = \mathcal{I},$$

then we split the integral of the right hand side into three parts as follows:

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 = \int_0^{\varepsilon_e^\delta} + \int_{\varepsilon_e^\delta}^{1/\varepsilon_e^\delta} + \int_{1/\varepsilon_e^\delta}^{+\infty},$$

where δ is a positive constant that will be chosen below. We have

$$\mathcal{I}_1 \le M\varepsilon_e^\gamma \int_0^{\varepsilon_e^\delta} \frac{w}{\varepsilon_e^\gamma} dw = \frac{M}{2}\,\varepsilon_e^{2\delta};$$

$$\mathcal{I}_3 \le M\varepsilon_e^\gamma \int_{1/\varepsilon_e^\delta}^{+\infty} \frac{w}{(\varepsilon_e w^2)^\gamma} dw = \frac{M}{2\gamma - 2}\,\varepsilon_e^{2\delta(\gamma-1)}.$$

Next, if $\gamma \neq 2$, we have

$$\mathcal{I}_2 \le M\varepsilon_e^\gamma \int_{\varepsilon_e^\delta}^{1/\varepsilon_e^\delta} \frac{w}{(2\sqrt{\varphi^\varepsilon}\, w)^\gamma} dw = \frac{M}{2\sqrt{\varphi^\varepsilon}} \left[\frac{\varepsilon_e^{\delta(\gamma-2)} - \varepsilon_e^{\delta(2-\gamma)}}{2-\gamma}\right] \varepsilon_e^\gamma.$$

Hence, in this case, it suffices to choose

$$\delta = \frac{1}{2}\left|\frac{\gamma}{\gamma - 2}\right|$$

to get the estimate

$$\mathcal{I}_2 \le \frac{M}{\sqrt{2\alpha}|\gamma - 2|}\,\varepsilon_e^{\gamma/2}.$$

In the case $\gamma = 2$, any δ is convenient since this estimate becomes

$$\mathcal{I}_2 \le \frac{M}{\sqrt{\varphi^\varepsilon}}\,\varepsilon_e^\gamma\,\delta\,|\mathrm{Log}\,\varepsilon_e|\,.$$

We choose for instance $\delta = 1$. Finally, as $\varepsilon_e \to 0$ and for every $\gamma > 3/2$, we have $\mathcal{I}_1 \to 0$, $\mathcal{I}_2 \to 0$ and $\mathcal{I}_3 \to 0$, thus

$$\left| j_e^{\varepsilon,+} - \sqrt{\varphi^\varepsilon} \int_0^{+\infty} f_e^\varepsilon dv \right| \to 0 \quad \text{uniformly on I.}$$

Clearly the same estimate holds for $-j_e^{\varepsilon,-}$:

$$\left| j_e^{\varepsilon,-} + \sqrt{\varphi^\varepsilon} \int_{-\infty}^{0} f_e^\varepsilon dv \right| \to 0 \quad \text{uniformly on I.}$$

Summing these two results, we deduce that n_e^ε converges uniformly to $\dfrac{j_{e,\mathrm{I}}^+ - j_{e,\mathrm{I}}^-}{\sqrt{\varphi(x)}}$ on I, which yields (3.29). Finally, (3.30) is a consequence of (3.29) and (3.9). □

We have the similar result for the ions:

Corollary 3.7 *Let* $\mathrm{I} = [a, b] \subset [0, 1]$ *such that* $\varphi(x) < 1$ *on* I. *Then, for* ε *small enough, we have the three equalities :*

$$\frac{dj_i^{\varepsilon,+}}{dx} \equiv \frac{dj_i^{\varepsilon,-}}{dx} \equiv 0 \quad \text{on I}$$

and

$$j_i^{\varepsilon,+} \to j_{i,\mathrm{I}}^+ \geq 0, \quad j_i^{\varepsilon,-} \to j_{i,\mathrm{I}}^- \leq 0 \text{ as } \varepsilon \to 0, \tag{3.31}$$

$j_{i,\mathrm{I}}^+$ *and* $j_{i,\mathrm{I}}^-$ *being two constant real numbers. Moreover* n_i^ε *and* k_i^ε *converge uniformly to* n_i *and* k_i *on* I *and, for* $x \in \mathrm{I}$, *we have*

$$j_i = j_{i,\mathrm{I}}^+ + j_{i,\mathrm{I}}^-, \tag{3.32}$$

$$n_i(x) = \frac{j_{i,\mathrm{I}}^+ - j_{i,\mathrm{I}}^-}{\sqrt{1 - \varphi(x)}}, \tag{3.33}$$

$$k_i(x) = (j_{i,\mathrm{I}}^+ - j_{i,\mathrm{I}}^-)\sqrt{1 - \varphi(x)}. \tag{3.34}$$

We now prove that the currents of electrons and ions do not vanish in the diode, at the limit $\varepsilon \to 0$. This result is physically natural, but not obvious mathematically.

Lemma 3.8 *Let* $j_e^g > 0$ *and* $j_i^g < 0$. *Then* $j_e > 0$ *and* $j_i < 0$.

Proof. We already know, by (2.26), that $j_e \geq 0$ and $j_i \leq 0$. Let us show that $j_i < 0$ using a contradiction argument; in the sequel of this proof we assume $j_i = 0$.

Set $a = \min\{x \in [0, 1] : \varphi(x) = 1/2\}$ and denote I=$[0, a]$. For all $x \in \mathrm{I}$ we have $\varphi(x) \leq 1/2$. Hence Corollary 3.7 implies

$$j_i = j_{i,\mathrm{I}}^+ + j_{i,\mathrm{I}}^-, \quad k_i(x) = (j_{i,\mathrm{I}}^+ - j_{i,\mathrm{I}}^-)\sqrt{1 - \varphi(x)}$$

on this interval I. Moreover, the boundary condition (2.18) gives $j_i^{\varepsilon,+}(0) = 0$ for all ε, so $j_{i,\mathrm{I}}^+ = 0$. Thus, with the assumption $j_i = 0$, it yields $j_{i,\mathrm{I}}^- = j_i - j_{i,\mathrm{I}}^+ = 0$ and

$$k_i \equiv 0 \text{ on } [0, a]. \tag{3.35}$$

Write now (2.23) at the point 0. By (3.11) it comes

$$h^\varepsilon - k_i^\varepsilon(0) = k_e^\varepsilon(0) - \frac{1}{4}\left(\frac{d\varphi^\varepsilon}{dx}(0)\right)^2 \leq k_e^\varepsilon(0) \leq C\varepsilon_e;$$

therefore, after passing to the limit and by (3.35), we have

$$h \leq 0. \tag{3.36}$$

We shall now prove that $h = 0$. Let $0 < \eta < 1/2$ and $\tilde{\mathrm{I}}=]b_1, b_2[$ be the maximal open interval containing a such that $\varphi > \eta$ on $\tilde{\mathrm{I}}$. Thanks to Lemma 3.6, we have, for some nonnegative constant C_0,

$$k_e = C_0\sqrt{\varphi}, \quad \text{on } \tilde{\mathrm{I}}. \tag{3.37}$$

Assume that $\frac{d\varphi^\varepsilon}{dx}$ does not vanish on $\tilde{\mathrm{I}}$. From the definitions of a and $\tilde{\mathrm{I}}$, we first deduce that $b_2 = 1$ and that $\frac{d\varphi^\varepsilon}{dx} > 0$ on $\tilde{\mathrm{I}}$. Next we construct a C^∞ test function $\theta_\delta(v)$ that verifies, for $\delta > 0$,

$$\begin{cases} \theta_\delta(v) = 1 \quad \text{for } v \leq -\delta \\ 0 < \theta_\delta(v) < 1 \quad \text{and} \quad \theta_\delta'(v) < 0 \quad \text{for } v \in (-\delta, 0) \\ \theta_\delta(v) = 0 \quad \text{for } v \geq 0 \end{cases}$$

Plugging this test function in a weak formulation of the Vlasov equation for the ions on the set $[b_1, 1] \times \mathbb{R}$, we get

$$\int_{b_1}^1 \int_{-\infty}^{+\infty} \left(v\frac{\partial\theta_\delta}{\partial x} - \frac{1}{2}\frac{d\varphi^\varepsilon}{dx}\frac{\partial\theta_\delta}{\partial v} \right) f_i^\varepsilon \, dxdv = \int_{-\infty}^{+\infty} v\theta_\delta \, f_i^\varepsilon(1, v)dv - \int_{-\infty}^{+\infty} v\theta_\delta \, f_i^\varepsilon(b_1, v)dv.$$

Hence, using the properties of θ_δ and the boundary condition (2.19), we deduce

$$\int_{-\infty}^0 vg_i^\varepsilon(v)\theta_\delta(v)dv - \int_{-\infty}^0 v\theta_\delta(v)f_i^\varepsilon(b_1, v)dv = -\frac{1}{2}\int_{b_1}^1 \int_{-\infty}^{+\infty} \frac{d\varphi^\varepsilon}{dx}\frac{d\theta_\delta}{dv} f_i^\varepsilon \, dxdv. \tag{3.38}$$

In the left hand side we have

$$\left| \int_{-\infty}^0 v\theta_\delta(v)f_i^\varepsilon(b_1, v)dv \right| \leq j_i^{\varepsilon,-}(b_1) \underset{\varepsilon\to 0}{\longrightarrow} j_{i,\mathrm{I}}^- = 0$$

and

$$\int_{-\infty}^0 vg_i^\varepsilon(v)\theta_\delta(v)dv \underset{\varepsilon\to 0}{\longrightarrow} j_i^g < 0.$$

Therefore, the left hand side of (3.38) is negative for ε, δ small enough. Besides, thanks to $\theta_\delta' \leq 0$ and the assumption $\frac{d\varphi^\varepsilon}{dx} > 0$, the right hand side of this equation is nonnegative. Hence (3.38) leads to a contradiction : there exists a point $x_1^\varepsilon \in \tilde{\mathrm{I}}$ such that $\frac{d\varphi^\varepsilon}{dx}(x_1^\varepsilon) = 0$. At this point, (2.23) reads

$$h^\varepsilon = k_e^\varepsilon(x_1^\varepsilon) + k_i^\varepsilon(x_1^\varepsilon) \geq 0. \tag{3.39}$$

Pass to the limit in this equality. We obtain $h \geq 0$, then, with (3.36),

$$h = 0. \tag{3.40}$$

Now we recall that $\varphi > \eta$ on $\tilde{\mathrm{I}}$, and that, by Lemma 3.6, k_e^ε converges uniformly to k_e which is given by (3.37). Let us show that $C_0 = 0$. Indeed, if $C_0 > 0$, then for ε small enough we have $k_e^\varepsilon > \sqrt{\eta} C_0/2$ on $\tilde{\mathrm{I}}$. Therefore (3.39) leads to

$$\sqrt{\eta} C_0/2 < h^\varepsilon,$$

which implies, by (3.40), that $C_0 \le 0$. Hence $C_0 = 0$ and (3.37) becomes

$$k_e \equiv 0 \text{ on }]b_1, b_2[. \tag{3.41}$$

Finally, from (3.35), (3.40), (3.37) and (3.26), we deduce that

$$\frac{d\varphi}{dx} \equiv 0 \text{ on }]b_1, a[.$$

This is in contradiction with $\varphi(b_1) = \eta < 1/2$ and $\varphi(a) = 1/2$. Hence $j_i < 0$. The property $j_e > 0$ is obtained in the same way. □

In the sequel, we shall always assume $j_e^g > 0$ and $j_i^g < 0$.

Lemma 3.9 *For ε small enough, the densities n_e^ε and n_i^ε verify the estimates*

$$\frac{j_e^\varepsilon}{2\sqrt{\varphi^\varepsilon + C\varepsilon_e^\beta}} \le n_e^\varepsilon, \tag{3.42}$$

$$\frac{-j_i^\varepsilon}{2\sqrt{1 - \varphi^\varepsilon + C\varepsilon_i^\beta}} \le n_i^\varepsilon, \tag{3.43}$$

where β is the same positive real number as in Lemma 3.3.

Proof. We only show the estimate concerning the electrons (3.42). The starting point of this proof is the same as in [2], Corollary 2.4. Thanks to (3.13) we have $-C\varepsilon_e^\beta \le \varphi^\varepsilon$. Hence

$$\begin{aligned} n_e^\varepsilon &\ge \int_0^{+\infty} f_e^\varepsilon(x,v)dv \\ &\ge \int_0^{\sqrt{\varphi^\varepsilon + C\varepsilon_e^\beta}} f_e^\varepsilon(x,v)dv \\ &\ge \frac{1}{\sqrt{\varphi^\varepsilon + C\varepsilon_e^\beta}} \int_0^{\sqrt{\varphi^\varepsilon + C\varepsilon_e^\beta}} v f_e^\varepsilon(x,v)dv \\ &\ge \frac{j_e^\varepsilon}{\sqrt{\varphi^\varepsilon + C\varepsilon_e^\beta}} - \frac{\mathcal{R}_\varepsilon}{\sqrt{\varphi^\varepsilon + C\varepsilon_e^\beta}}, \end{aligned}$$

where

$$\mathcal{R}_\varepsilon = \int_{\sqrt{\varphi^\varepsilon + C\varepsilon_e^\beta}}^{+\infty} v f_e^\varepsilon(x,v)dv.$$

It remains to prove that, for ε small enough, $\mathcal{R}_\varepsilon \leq j_e^\varepsilon/2$. For that we use (3.3)

$$\mathcal{R}_\varepsilon \leq \frac{M}{\varepsilon_e^2} \int_{\sqrt{\varphi^\varepsilon + C\varepsilon_e^\beta}}^{+\infty} \frac{v}{\left(1 + \dfrac{v^2 - \varphi^\varepsilon}{\varepsilon_e^2}\right)^\gamma} dv.$$

Let us apply the change of variable $w = \dfrac{\sqrt{v^2 - \varphi^\varepsilon - C\varepsilon_e^\beta}}{\varepsilon_e}$. We have

$$\begin{aligned}
\mathcal{R}_\varepsilon &\leq M \int_0^{+\infty} \frac{w}{\left(1 + w^2 + C\varepsilon_e^{\beta-2}\right)^\gamma} dw \\
&\leq \frac{M}{\sqrt{C}}\ \varepsilon_e^{1-\beta/2} \int_0^{+\infty} \frac{w}{(1+w^2)^{\gamma-1/2}} dw \\
&= \frac{M}{\sqrt{C}(2\gamma - 3)}\ \varepsilon_e^{1-\beta/2}.
\end{aligned}$$

Recall that $\gamma > 3/2$ and $1 - \beta/2 = \dfrac{3}{2\gamma+1} > 0$. Since with Lemma 3.8, $j_e > 0$, we conclude that, for ε small enough, $\mathcal{R}_\varepsilon < j_e^\varepsilon/2$, or $j_e^\varepsilon - \mathcal{R}_\varepsilon > j_e^\varepsilon/2$. □

Lemma 3.10 *The set $\{x \in [0,1], \varphi(x) = 0$ or $\varphi(x) = 1\}$ is finite.*

Proof. This proof is adapted from those of Lemmas 3.1 and 4.1 of [2].

Let $r_1 < r_2$ be two real numbers such as $\varphi(r_1) = \varphi(r_2) = 0$. We will find a positive lower bound for $r_2 - r_1$, which is sufficient to conclude. Set $s^\varepsilon = \max(\varphi^\varepsilon(r_1), \varphi^\varepsilon(r_2))$. We have $s^\varepsilon \to 0$ as $\varepsilon \to 0$.

Moreover we have $\|\dfrac{d\varphi^\varepsilon}{dx}\|_{L^\infty(0,1)} \leq C_0$. Since we only want a lower bound for $r_2 - r_1$, without loss of generality we can assume $r_2 - r_1 < 1/C_0$. Hence, if ε is small enough,

$$\forall x \in [r_1, r_2] \quad \varphi^\varepsilon(x) < 1/2 + s^\varepsilon < 3/4.$$

From (3.8) it follows

$$n_i^\varepsilon \leq n_D, \tag{3.44}$$

where n_D is a constant positive real number. We now define

$$\mu^\varepsilon = -\inf_{[r_1,r_2]} \left(\frac{d^2\varphi^\varepsilon}{dx^2}\right).$$

The maximum principle implies

$$\forall x \in [r_1, r_2], \quad \varphi^\varepsilon(x) \begin{aligned}[t] &\le \mu^\varepsilon \frac{(x-r_1)(r_2-x)}{2} + \max(\varphi^\varepsilon(r_1), \varphi^\varepsilon(r_2)) \\ &\le \mu^\varepsilon \frac{(r_2-r_1)^2}{8} + s^\varepsilon. \end{aligned}$$

By (3.13), we have

$$\mu^\varepsilon \frac{(r_2-r_1)^2}{8} + s^\varepsilon \ge -C\varepsilon_e^\beta,$$

and by (3.44), (3.42) and Poisson's equation (2.14),

$$\frac{d^2\varphi^\varepsilon}{dx^2} \ge \frac{j_e^\varepsilon}{2\sqrt{\mu^\varepsilon \frac{(r_2-r_1)^2}{8} + s^\varepsilon + C\varepsilon_e^\beta}} - n_D \text{ on } [r_1, r_2],$$

which leads, with the definition of μ^ε as a lower bound, to

$$\mu^\varepsilon \le n_D - \frac{j_e^\varepsilon}{2\sqrt{\mu^\varepsilon \frac{(r_2-r_1)^2}{8} + s^\varepsilon + C\varepsilon_e^\beta}}.$$

One can see that for ε small enough this implies the existence of a positive solution to the following equation in ξ:

$$\xi = n_D - \frac{j_e^\varepsilon}{2\sqrt{\xi \frac{(r_2-r_1)^2}{8} + s^\varepsilon + C\varepsilon_e^\beta}}.$$

Set $x^2 = \xi \dfrac{(r_2-r_1)^2}{8} + s^\varepsilon + C\varepsilon_e^\beta$; then x satisfies

$$x^3 - \left(\frac{(r_2-r_1)^2}{8} n_D + s^\varepsilon + C\varepsilon_e^\beta \right) x + \frac{j_e^\varepsilon (r_2-r_1)^2}{16} = 0.$$

As in [2], Lemma 3.1, straightforward calculations show that this equation has a positive solution if and only if

$$\frac{j_e^\varepsilon}{2} \le \frac{1}{3\sqrt{6}} (r_2-r_1) n_D^{3/2} \left[1 + \frac{8}{n_D (r_2-r_1)^2} (s^\varepsilon + C\varepsilon_e^\beta) \right].$$

Hence, by passing to the limit,

$$r_2 - r_1 \ge \frac{3\sqrt{6}\, j_e}{2\, n_D^{3/2}}.$$

It gives a positive lower bound for $r_2 - r_1$, thanks to Lemma 3.8. The same result can be obtained symmetrically for two points where $\varphi = 1$. □

Applying Lemma 3.10, one can find an integer m and $m+2$ real numbers $0 = r_0 < r_1 < ... < r_m < r_{m+1} = 1$ such that

$$\forall k \in \{0,..,m+1\} \quad \varphi(r_k) = 0 \text{ or } \varphi(r_k) = 1$$

and

$$\forall k \in \{0,..,m\} \quad \forall x \in]r_k, r_{k+1}[\quad 0 < \varphi(x) < 1.$$

Then, by Lemma 3.6 and its Corollary 3.7, the positive and negative currents of electrons and ions $j_e^+, j_e^-, j_i^+, j_i^-$ are constant on each one of these intervals $]r_k, r_{k+1}[$. We denote these numbers with the index k, and set

$$\overline{J}_{e,k} = j_{e,k}^+ - j_{e,k}^- \geq 0, \quad \overline{J}_{i,k} = -j_{i,k}^- + j_{i,k}^+ \geq 0, \tag{3.45}$$

For $k \in \{0,..,m\}$, $x \in]r_k, r_{k+1}[$, we have

$$j_e = j_{e,k}^+ + j_{e,k}^-, \quad j_i = j_{i,k}^- + j_{i,k}^+, \tag{3.46}$$

$$n_e(x) = \overline{J}_{e,k}/\sqrt{\varphi(x)}, \quad n_i(x) = \overline{J}_{i,k}/\sqrt{1-\varphi(x)}, \tag{3.47}$$

$$k_e(x) = \overline{J}_{e,k}\sqrt{\varphi(x)}, \quad k_i(x) = \overline{J}_{i,k}\sqrt{1-\varphi(x)}. \tag{3.48}$$

The following result will be crucial for the proof of (2.29) and will enable us further to precise the form of the limit potential.

Lemma 3.11 *The integer m above defined is even, denoted $2p$. Moreover, for every $k \in \{0,..,2p\}$, the potential φ is monotone on $[r_k, r_{k+1}]$ and we have*

$$j_{e,k}^- = 0, \quad j_{i,k}^+ = 0. \tag{3.49}$$

Proof. The fact that m is even will be a direct consequence of the monotonicity of φ on each interval $[r_k, r_{k+1}]$. Indeed, to verify the boundary conditions for the potential (2.20), we must have $m = 2p$, and

$$\forall 0 \leq k \leq p \quad \varphi(r_{2k}) = 0, \quad \varphi(r_{2k+1}) = 1.$$

Remark now that the boundary conditions (2.17) and (2.18) imply

$$j_{e,m}^- = 0 \text{ and } j_{i,0}^+ = 0; \tag{3.50}$$

if $m=0$ then (3.49) is already proved. Moreover, by (3.45), (3.46) and (3.48) we have

$$\text{for } x\in[0,r_1],\quad k_i(x)=-j_i\sqrt{1-\varphi(x)},$$

$$\text{for } x\in[r_m,1],\quad k_e(x)=j_e\sqrt{\varphi(x)}.$$

Hence, if we write (3.26) at the points 0 and 1, thanks to (3.11) and (3.12) we obtain

$$h\le -j_i \quad\text{and}\quad h\le j_e. \tag{3.51}$$

Assume now $m\ge 1$. Let $0<r_k<1$ and $\varphi(r_k)=0$. Corollary 3.7 already implies $j^+_{i,k-1}=j^+_{i,k}$ and $j^-_{i,k-1}=j^-_{i,k}$. We shall prove that in fact we have

$$j^+_{i,k-1}=j^+_{i,k}=0,\quad j^-_{i,k-1}=j^-_{i,k}=-h. \tag{3.52}$$

By Lemma 3.10, r_k is an isolated zero of the nonnegative function φ. Let $0<\eta<1/2$ and $b_1<r_k<b_2$ be such that $\varphi(b_1)=\varphi(b_2)=\eta$ and $\varphi<\eta$ on the interval $]b_1,b_2[$. Next we define x^ε by $\varphi^\varepsilon(x^\varepsilon)=\min\limits_{[b_1,b_2]}\varphi$. Since $\varphi(r_k)=0$, we have $\varphi^\varepsilon(r_k)<\eta$ for ε small enough. Therefore this point x^ε is located inside $]b_1,b_2[$ and it is easy to see that

$$\frac{d\varphi^\varepsilon}{dx}(x^\varepsilon)=0\quad\text{and } x^\varepsilon\to r_k \text{ as } \varepsilon\to 0.$$

Hence, if we write (2.23) at this point x^ε, we obtain

$$h^\varepsilon=k_i^\varepsilon(x^\varepsilon)+k_e^\varepsilon(x^\varepsilon). \tag{3.53}$$

By Corollary 3.7 and (3.11), we have $k_i^\varepsilon(x^\varepsilon)\to\overline{J}_{i,k}$ and $k_e^\varepsilon(x^\varepsilon)\to 0$. Therefore, after passing to the limit in (3.53), we have $h=\overline{J}_{i,k}$. Thus (3.51) implies $\overline{J}_{i,k}\le -j_i$ and finally, with (3.45) and (3.46) we deduce (3.52). We have proved that, if $m\ge 1$,

$$\left\{\begin{array}{l} j^+_{i,0}=0\\ \text{for } 1\le k\le m,\quad \varphi(r_k)=0\Longrightarrow\left\{\begin{array}{l} j^+_{i,k}=j^+_{i,k-1}=0\\ j^-_{i,k}=j^-_{i,k-1}=-h.\end{array}\right.\end{array}\right. \tag{3.54}$$

Similarly, we have the corresponding result for the electrons

$$\left\{\begin{array}{l} j^-_{e,m}=0\\ \text{for } 1\le k\le m,\quad \varphi(r_k)=1\Longrightarrow\left\{\begin{array}{l} j^-_{e,k}=j^-_{e,k-1}=0\\ j^+_{e,k}=j^+_{e,k-1}=h.\end{array}\right.\end{array}\right. \tag{3.55}$$

To end the proof of the lemma, it is sufficient now to prove that φ is monotone on each $[r_k,r_{k+1}]$. Indeed it will imply that φ takes alternately the values 0 and 1 on the sequence $(r_k)_{0\le k\le m}$, then, by (3.54), (3.55), that for every $0\le k\le m$, there holds $j^-_{e,k}=j^+_{i,k}=0$.

We now consider the most general case $m\ge 0$ and use a contradiction argument. Assume that φ is not monotone on $[r_k,r_{k+1}]$, $0\le k\le m$. We shall only treat the case where $\varphi(r_k) = 0$, the case $\varphi(r_k) = 1$ can be treated similarly. By (3.50), (3.51), (3.54) and the definitions (3.45) of $\overline{J}_{e,k}$, $\overline{J}_{i,k}$, we have

$$j_{i,k}^+ = 0, \quad h \leq -j_i = \overline{J}_{i,k}, \quad h \leq j_e \leq \overline{J}_{e,k}. \tag{3.56}$$

Instead of $[r_k, r_{k+1}]$, we can consider a positive real number δ and a smaller interval, $[r_k + \delta, r_{k+1} - \delta]$, where φ is not monotone. Next, for ε small enough, the C^1 function φ^ε is also non monotone on the same interval and there exists $x_0^\varepsilon \in [r_k + \delta, r_{k+1} - \delta]$ such that

$$\frac{d\varphi^\varepsilon}{dx}(x_0^\varepsilon) = 0.$$

At this point (2.23) writes $h^\varepsilon = k_e^\varepsilon(x_0^\varepsilon) + k_i^\varepsilon(x_0^\varepsilon)$. Moreover, after extraction of a subsequence, x_0^ε converges to $x_0 \in [r_k + \delta, r_{k+1} - \delta]$ and we have, after passing to the limit and by (3.48),

$$h = \overline{J}_{e,k}\sqrt{\varphi(x_0)} + \overline{J}_{i,k}\sqrt{1 - \varphi(x_0)}. \tag{3.57}$$

Let us show that this relation (3.57) implies $\overline{J}_{e,k} < h$, which leads to a contradiction with (3.56) and enables to conclude the proof of the lemma. Since $r_k < x_0 < r_{k+1}$, we have $0 < \varphi(x_0) < 1$. Therefore, if we set $\sqrt{\varphi(x_0)} = \sin\theta$, we have $\theta \in]0, \pi/2[$. (3.57) can then be rewritten as follows:

$$\overline{J}_{e,k} \sin\theta = h - \overline{J}_{i,k}\cos\theta;$$

thus (3.56) yields

$$\overline{J}_{e,k} \sin\theta \leq h(1 - \cos\theta),$$

and, since $\theta \in]0, \pi/2[$,

$$\overline{J}_{e,k} \leq h\tan(\theta/2) < h.$$

□

Proof of Theorem 2.3.

Since the number of points where φ takes the value 0 or 1 is finite, the uniform convergence of φ^ε and the estimates (3.11), (3.12) enable to complete on the whole interval $[0, 1]$ the convergence results of k_e^ε and k_i^ε stated in Lemma 3.6 and Corollary 3.7. Hence, by Lemma 3.11, we deduce that

$$k_e^\varepsilon \to k_e = j_e\ \sqrt{\varphi}, \quad k_i^\varepsilon \to k_i = -j_i\ \sqrt{1 - \varphi}, \tag{3.58}$$

uniformly on $[0, 1]$.

The crucial steps of the proof of Theorem 2.3 are the C^1 convergence of the potential and the L^1 convergence of the densities.

$C^1([0, 1])$ *convergence of* φ^ε.
We use the same method as [2], [4], splitting $[0, 1]$ into several intervals.

By Lemma 3.6 and Corollary 3.7, $n_e^\varepsilon \to n_e$ and $n_i^\varepsilon \to n_i$ in the L^∞ norm on each interval $[r_k + \alpha, r_{k+1} - \alpha]$, $\quad 0 \leq k \leq 2p, \quad \alpha > 0$. Thus $\varphi^\varepsilon \to \varphi$ in $W^{2,\infty}$ strong, and in particular in $C^1([r_k+\alpha, r_{k+1}-\alpha])$. We also infer that $\frac{d\varphi^\varepsilon}{dx}$ converges pointwise to the continuous function $\frac{d\varphi}{dx}$ on each $]r_k, r_{k+1}[$.

We now choose a particular α as follows. By (3.42), (3.8) and the uniform convergence of φ^ε, we have $n_e^\varepsilon(x) \geq n_i^\varepsilon(x)$ in a neighbourhood of each point r_{2k} where φ vanishes; such a neighbourhood can be chosen uniformly with respect to ε. Similarly, (3.43) and (3.6) allow to write $n_i^\varepsilon(x) \geq n_e^\varepsilon(x)$ uniformly in a neighbourhood of each point r_{2k+1}. For $\alpha > 0$, denote

$$V_0^\alpha = [0,\alpha],\ V_{2p+1}^\alpha = [1-\alpha, 1] \text{ and } V_k^\alpha = [r_k - \alpha, r_k + \alpha] \text{ for } 0 < k < 2p+1.$$

Thanks to Poisson's equation (2.14), it is thus possible to choose α small enough to ensure

$$\begin{cases} \dfrac{d\varphi^\varepsilon}{dx} \text{ is increasing on } V_{2k}^\alpha,\ 0 \leq k \leq p, \\ \dfrac{d\varphi^\varepsilon}{dx} \text{ is decreasing on } V_{2k+1}^\alpha,\ 0 \leq k \leq p. \end{cases} \tag{3.59}$$

Assume that

$$\lim_{x\to r_k} \frac{d\varphi}{dx}(x) = \lim_{\varepsilon\to 0} \frac{d\varphi^\varepsilon}{dx}(r_k) \text{ for } 0 \leq k \leq 2p+1. \tag{3.60}$$

We deduce that $\varphi^{\varepsilon\prime}$ converges pointwise on $[0,1]$ to the continuous function φ'. Hence, thanks to (3.59), Dini's theorem (we use here a version of Dini's theorem which deals with sequences of increasing functions) allows to conclude that $\varphi^{\varepsilon\prime}$ converges to φ' in the $C^0(V_k^\alpha)$ norm for $0 \leq k \leq 2p+1$. Since the number of points r_k is finite, this implies the C^1 convergence of φ^ε on $[0,1]$. It now remains to prove (3.60).

Thanks to the symmetry of the problem, we only consider r_{2k}, the points where φ vanishes. By (3.59), one can define

$$\varphi'((r_{2k})+) = \lim_{x\to(r_{2k})+} \varphi'(x), \quad \varphi'((r_{2k})-) = \lim_{x\to(r_{2k})-} \varphi'(x), \quad 0 < k \leq p,$$

$$\varphi'(0) = \lim_{x\to 0+} \varphi'(x).$$

Moreover, (2.23), (3.26) and (3.58) imply

$$\left|\frac{d\varphi}{dx}((r_{2k})+)\right| = \left|\frac{d\varphi}{dx}((r_{2k})-)\right| = \lim_{\varepsilon\to 0} \left|\frac{d\varphi^\varepsilon}{dx}(r_{2k})\right| = L = 2\sqrt{-j_i - h}. \tag{3.61}$$

Two cases can occur.

• If $p > 0$ then (3.54) holds and implies $h = -j_i$, so $L = 0$ and (3.60) is proved, since all the different limit values in (3.61) vanish. The result is the same if $p = 0$ and $L = 0$.

• If $p = 0$ and $L > 0$, since $\varphi \geq 0$ we have $\dfrac{d\varphi}{dx}(0+) = L > 0$. Assume that $\lim\limits_{\varepsilon\to 0} \dfrac{d\varphi^\varepsilon}{dx}(0) = -L$ and let x^ε be such that $\varphi^\varepsilon(x^\varepsilon) = \min\limits_{[0,1]} \varphi^\varepsilon$. For ε small enough, this assumption implies

$$\varphi^\varepsilon(x^\varepsilon) < 0 \text{ and } 0 < x^\varepsilon < 1.$$

Hence we have $\dfrac{d\varphi^\varepsilon}{dx}(x^\varepsilon) = 0$, then $-h^\varepsilon - k_i^\varepsilon(x) = -k_e^\varepsilon(x)$. By (3.58) and as $\varepsilon \to 0$, it yields $L = 0$, which leads to a contradiction. Thus $\lim_{\varepsilon\to 0} \dfrac{d\varphi^\varepsilon}{dx}(0) = L = \dfrac{d\varphi}{dx}(0+)$, which implies (3.60).

$L^1([0,1])$ *strong convergence of* n_e^ε *and* n_i^ε.
We show that n_i^ε is a Cauchy sequence in $L^1(0,1)$. Let α be an arbitrary small positive real number. Recall that the points where φ takes the value 1 are $(r_{2k+1})_{0\le k\le p}$. Similarly to the above defined intervals V_k^α, we denote

$$W_0^\alpha = [0, r_1 - \alpha] \text{ and } W_k^\alpha = [r_{2k-1} + \alpha, r_{2k+1} - \alpha] \text{ for } 0 < k \le p.$$

Next we write

$$\|n_i^{\varepsilon_1} - n_i^{\varepsilon_2}\|_{L^1} = \sum_{k=0}^{p} \int_{W_k^\alpha} |n_i^{\varepsilon_1} - n_i^{\varepsilon_2}| + \sum_{k=0}^{p} \int_{V_{2k+1}^\alpha} |n_i^{\varepsilon_1} - n_i^{\varepsilon_2}| = \mathcal{I}_1 + \mathcal{I}_2.$$

We first treat the term $\mathcal{I}_2$:

$$\mathcal{I}_2 \le \sum_{k=0}^{p} \int_{V_{2k+1}^\alpha} n_i^{\varepsilon_1} + \sum_{k=0}^{p} \int_{V_{2k+1}^\alpha} n_i^{\varepsilon_2};$$

we estimate these terms as for the proof of (3.15), writing

$$\begin{aligned} 0 \le \int_{y_1}^{y_2} n_i^\varepsilon(x)dx &= \int_{y_1}^{y_2} (n_e^\varepsilon \varphi^\varepsilon + n_i^\varepsilon(1-\varphi^\varepsilon)) - \int_{y_1}^{y_2} \frac{d^2\varphi^\varepsilon}{dx^2}\varphi^\varepsilon \\ &= \int_{y_1}^{y_2} (n_e^\varepsilon \varphi^\varepsilon + n_i^\varepsilon(1-\varphi^\varepsilon)) + \int_{y_1}^{y_2} (\frac{d\varphi^\varepsilon}{dx})^2 \\ &\quad - \frac{d\varphi^\varepsilon}{dx}(y_2)\varphi^\varepsilon(y_2) + \frac{d\varphi^\varepsilon}{dx}(y_1)\varphi^\varepsilon(y_1) \\ &\le (C_1 + C_2)(y_2 - y_1) - \frac{d\varphi^\varepsilon}{dx}(y_2)\varphi^\varepsilon(y_2) + \frac{d\varphi^\varepsilon}{dx}(y_1)\varphi^\varepsilon(y_1), \end{aligned}$$

where we used (3.20) and (3.14). The right hand side of this inequality can be chosen arbitrary small by taking $|y_2 - y_1|$ and ε small enough, thanks to the $C^1([0,1])$ convergence of φ^ε. Remark that the width of V_k^α is bounded by 2α. Thus, for a given $\eta > 0$, we can choose α such that, for $\max(\varepsilon_1, \varepsilon_2)$ small enough, we have $\mathcal{I}_2 < \eta/2$.

The real α being fixed, the term $\mathcal{I}_1$ can now be estimated. Indeed, for all $x \in W_k^\alpha$, we have

$$\inf_{0\le k\le p} |x - r_{2k+1}| \ge \alpha,$$

which implies, by Corollary 3.7, that the sequence n_i^ε converges to n_i in $L^\infty(W_k^\alpha)$ strong. Hence, for $\max(\varepsilon_1, \varepsilon_2)$ small enough, we also have $\mathcal{I}_1 < \eta/2$. This allows us to conclude finally

$$n_i^\varepsilon \to n_i \text{ in } L^1(0,1) \text{ strong.}$$

Similarly, one can prove that

$$n_e^\varepsilon \to n_e \text{ in } L^1(0,1) \text{ strong.}$$

This implies

$$\varphi^\varepsilon \to \varphi \text{ in } W^{2,1}([0,1]) \text{ strong.}$$

The first part of Theorem 2.3 is proved. Since $d\varphi^\varepsilon/dx$ converges in $C^0([0,1])$, we can pass to the limit in (2.12) and (2.13), and the limit distribution functions f_e, f_i are weak solutions of the Vlasov equations (2.30). Moreover, since we know that n_e and n_i are $L^1(0,1)$ functions, (2.28) can be deduced from (3.47). Hence (3.24) holds in the L^1 sense, instead of the sense of the bounded measures, and writes (2.31). Finally, from the support properties of f_e and f_i (3.22), (3.23), and from Lemma 3.11, we have

$$\text{Supp}(f_e) = \left\{(x,v)/\ 0 \le x \le 1 \ \text{ and } \quad v = \sqrt{\varphi(x)}\right\},$$
$$\text{Supp}(f_i) = \left\{(x,v)/\ 0 \le x \le 1 \ \text{ and } \quad v = -\sqrt{1-\varphi(x)}\right\},$$

which gives, with (3.25) and (2.28), the expressions (2.29) of f_e and f_i.

□

4. Analysis of the reduced problem

In this section we study the limit model. By Theorem 2.3, this model is completely characterized as soon as the currents $j_e > 0$, $j_i < 0$ and the potential $\varphi(x)$ are known. This section is divided into two parts. In the first one we characterize all the solutions such that the potential is increasing; they are described in Theorem 2.4. In the second part we treat the general case, and prove Theorems 2.4, 2.5.

Recall first what are the most general equations of the limit model, deduced from Section 3.. This model writes (2.31), (2.32), with $\varphi \in W^{2,1}([0,1])$. Moreover, there exists an integer p and $2p+2$ real numbers

$$0 = r_0 < ... < r_{2k} < r_{2k+1} < ... < r_{2p+1} = 1,$$

such that

$$\varphi(r_{2k}) = 0, \ \varphi(r_{2k+1}) = 1,$$
$$0 < \varphi(x) < 1 \text{ if } x \notin \{(r_k)_{0 \le k \le 2p+1}\}$$

and φ is monotone on each $]r_k, r_{k+1}[$. Recall also the first integral

$$h = j_e\sqrt{\varphi(x)} - j_i\sqrt{1-\varphi(x)} - \frac{1}{4}\left(\frac{d\varphi}{dx}(x)\right)^2. \tag{4.1}$$

Finally, to complete this model we add the following lemma.

Lemma 4.1 *We have* $0 < j_e \leq j_e^g$ *and* $0 < -j_i \leq -j_i^g$. *Moreover,*

$$\begin{cases} \text{if } \ \varphi'(0) > 0 & \text{then } j_e = j_e^g, \\ \text{if } \ \varphi'(1) > 0 & \text{then } j_i = j_i^g. \end{cases} \tag{4.2}$$

Proof. The first part of the lemma is immediate, by (2.26). Moreover, we do not show its second part, (4.2) : this result is standard and we refer to Lemma 4.6 of [9] or Lemma 3.5 of [2] for its proof (this proof is based on a weak formulation of Vlasov's equation, in which we plug the test function θ_δ already used in the proof of Lemma (3.8)). □

4.1. Increasing potential

In this subsection we assume $p = 0$. Thus $0 < \varphi(x) < 1$ for $x \in]0,1[$ and φ is increasing from 0 to 1.

Before solving the complete limit model, in a first step we only consider the first integral (4.1) and the condition $\varphi(0) = 0$. Moreover, the three real numbers $j_e > 0$, $j_i < 0$, h are given here and verify the necessary conditions $h \leq j_e$ and $h \leq -j_i$. Since the potential is increasing, (4.1) also writes

$$\frac{d\varphi}{dx}(x) = 2\sqrt{j_e\sqrt{\varphi(x)} - j_i\sqrt{1-\varphi(x)} - h} > 0, \quad \text{for } 0 < x < 1;$$

hence, after an integration we can express the variable x in function of φ, for $0 < \varphi < 1$:

$$\frac{1}{2}\int_0^\varphi \frac{du}{\sqrt{j_e\sqrt{u} - j_i\sqrt{1-u} - h}} = x. \tag{4.3}$$

For $a > 0$, $b > 0$, $c \leq \min(a,b)$, set

$$\mathcal{G}(a,b,c) = \frac{1}{2}\int_0^1 \frac{du}{\sqrt{a\sqrt{u} + b\sqrt{1-u} - c}}.$$

The formula (4.3) enables to define implicitly an increasing function $\varphi(x)$ on a maximal interval $[0, x_0]$, where x_0 is given by

$$x_0 = \mathcal{G}(j_e, -j_i, h).$$

Moreover, we have $\varphi(0) = 0$ and $\varphi(x_0) = 1$. Therefore, as soon as j_e, j_i and h are given, the increasing potential that verifies (4.1) and $\varphi(0) = 0$ is uniquely characterized. Clearly, a derivation of (4.1) shows that this potential verifies (2.31). The Dirichlet boundary conditions (2.32) are fulfilled if, and only if $x_0 = 1$.

Consequently, to solve completely the limit model in the case of an increasing potential, it now remains to determine all the sets of parameters j_e, j_i, h, verifying

$$h \leq j_e, \quad h \leq -j_i, \quad (4.2) \quad \text{and} \quad \mathcal{G}(j_e, -j_i, h) = 1. \tag{4.4}$$

Remark that, by (4.1),

$$\begin{cases} \varphi'(0) > 0 \Longleftrightarrow h < -j_i, \\ \varphi'(1) > 0 \Longleftrightarrow h < j_e. \end{cases} \tag{4.5}$$

Depending on the values of φ' in 0 and 1, by Lemma 4.1, four cases occur. In fact, these four cases will correspond exactly to the four cases of Theorem 2.4.

(i) $\{\varphi'(0) > 0 \text{ and } \varphi'(1) > 0\}$.
In this case, the currents j_e and j_i are respectively fixed equal to j_e^g and j_i^g. h is the only unknown, and, by (4.4) and (4.5), has to be chosen such that $\mathcal{G}(j_e^g, -j_i^g, h) = 1$ and $h < \min(j_e^g, -j_i^g)$. Let $F_1(h) = \mathcal{G}(j_e^g, -j_i^g, h)$. This function F_1 is defined on $]-\infty, \min(j_e^g, -j_i^g)]$, is increasing, and tends to 0 as $h \to -\infty$. Hence one can find a suitable h if and only if

$$\mathcal{G}(j_e^g, -j_i^g, \min(j_e^g, -j_i^g)) > 1. \tag{4.6}$$

Moreover, in that case h is unique. Consider the quarter-plane $(j_e^g, -j_i^g) \in (\mathbb{R}_+^*)^2$, and in this quarter-plane the two curves implicitly defined by

$$(\mathcal{C}_e) = \{\mathcal{G}(j_e^g, -j_i^g, -j_i^g) = 1, \quad j_e^g \geq -j_i^g\}$$

$$(\mathcal{C}_i) = \{\mathcal{G}(j_e^g, -j_i^g, j_e^g) = 1, \quad -j_i^g \geq j_e^g\}.$$

These curves are represented on Figure 1. Denote by $(j_2^{\mathrm{CL}}, j_2^{\mathrm{CL}})$ the point where they intersect. We have $\mathcal{G}(j_2^{\mathrm{CL}}, j_2^{\mathrm{CL}}, j_2^{\mathrm{CL}}) = 1$, thus

$$j_2^{\mathrm{CL}} = \frac{1}{2}\left(\int_0^1 \frac{du}{\sqrt{\sqrt{u} + \sqrt{1-u} - 1}}\right)^2.$$

It is easy to see that the condition (4.6) corresponds exactly to the domain (i) in Figure 1; the case (i) of Theorem 2.4 is proved.

(ii) $\{\varphi'(1) = 0 \text{ and } \varphi'(0) > 0\}$.
By (4.2) and (4.5), we have $h = j_e = j_e^g$. The only unknown, j_i, must verify $j_e^g < -j_i \leq -j_i^g$ and $\mathcal{G}(j_e^g, -j_i, j_e^g) = 1$. Let $F_2(j_i) = \mathcal{G}(j_e^g, -j_i, j_e^g)$. This function F_2 is decreasing. Thus, to find an adequate j_i, it is necessary and sufficient that

$$\begin{cases} -j_i^g > j_e^g, \\ \mathcal{G}(j_e^g, -j_i^g, j_e^g) \leq 1, \ . \\ \mathcal{G}(j_e^g, j_e^g, j_e^g) > 1 \end{cases} \tag{4.7}$$

The last one of these conditions also reads $j_e^g < j_2^{\mathrm{CL}}$. Therefore (4.7) is verified if and only if $(j_e^g, -j_i^g)$ is in the domain (ii) of Figure 1. Moreover, the current $-j_i$ is adjusted such that $(j_e^g, -j_i) \in (\mathcal{C}_i)$; the case (ii) of Theorem 2.4 is proved.

(iii) $\{\varphi'(1) > 0 \text{ and } \varphi'(0) = 0\}$.

Since the problem is symmetric, we find

$$\begin{cases} j_e^g > -j_i^g, \\ \mathcal{G}(j_e^g, -j_i^g, -j_i^g) \leq 1, \\ -j_e^g < j_2^{\mathrm{CL}}, \end{cases}$$

which corresponds exactly to the domain (iii).

(iv)$\{\varphi'(0) = 0$ and $\varphi'(1) = 0\}$.
By (4.5), we must have $h = j_e = -j_i$, so $\mathcal{G}(h, h, h) = 1$. Hence, if

$$j_e^g \geq j_2^{\mathrm{CL}}, \quad -j_i^g \geq j_2^{\mathrm{CL}},$$

then the solution is $h = j_e = -j_i = j_2^{\mathrm{CL}}$. This treats the case (iv) of Theorem 2.4. Recall that in this case the potential is denoted φ_{CL} and is represented on Figure 2.

4.2. General case

In this part, we do not make any assumption on the monotonicity of the potential and treat the general case. Consequently, we have an odd number of intervals where φ is monotone, either increasing from 0 to 1 or decreasing from 1 to 0.

Consider the case $p > 0$ and write (3.26) at the points r_k for $0 < k < 2p + 1$. This implies $j_e = -j_i = h$ and $\varphi'(r_k) = 0$.

On each $[r_{2k}, r_{2k+1}]$, the potential is increasing. Setting

$$\begin{cases} y = \dfrac{x - r_{2k}}{r_{2k+1} - r_{2k}}, \quad \psi(y) = \varphi(x), \\ \\ \tilde{j}_e = (r_{2k+1} - r_{2k})^2 j_e, \quad \tilde{j}_i = (r_{2k+1} - r_{2k})^2 j_i, \end{cases}$$

the function ψ verifies the system

$$\begin{cases} \psi \in W^{2,1}([0,1]), \\ \dfrac{d^2\psi}{dy^2}(y) = \dfrac{\tilde{j}_e}{\sqrt{\psi(y)}} + \dfrac{\tilde{j}_i}{\sqrt{1 - \psi(y)}}, \quad y \in [0,1], \\ \psi(0) = 0, \ \psi(1) = 1, \ \psi'(0) = \psi'(1) = 0. \end{cases}$$

It appears that ψ is exactly an increasing solution of (2.31) in the case *(iv)*. Therefore, from the previous subsection, we get

$$\psi \equiv \varphi_{\mathrm{CL}}, \quad \tilde{j}_e = -\tilde{j}_i = j_2^{\mathrm{CL}}.$$

This implies

$$j_e = -j_i = \frac{1}{(r_{2k+1} - r_{2k})^2}\, j_2^{\mathrm{CL}}, \quad \varphi(x) = \varphi_{\mathrm{CL}}\left(\frac{x - r_{2k}}{r_{2k+1} - r_{2k}}\right).$$

For the intervals $[r_{2k-1}, r_{2k}]$, where the potential is decreasing, a suitable rescaling allows to conclude that

$$j_e = -j_i = \frac{1}{(r_{2k} - r_{2k-1})^2}\, j_2^{\mathrm{CL}}, \quad \varphi(x) = \varphi_{\mathrm{CL}}\left(\frac{r_{2k} - x}{r_{2k} - r_{2k-1}}\right).$$

Finally, since the currents j_e and j_i are constant on $[0, 1]$, all these intervals are of the same width, $r_{k+1} - r_k = 1/(2p + 1)$, so

$$j_e = -j_i = (2p + 1)^2 j_2^{\mathrm{CL}}.$$

When p is given, the potential is completely characterized and takes the form described in Theorem (2.5).

Moreover Lemma 4.1 implies that p must be chosen such that

$$(2p + 1)^2 j_2^{\mathrm{CL}} \leq \min\{j_e^g, -j_i^g\}.$$

Hence, if $(\mathcal{H})$ is verified, then the only possibility is $p = 1$ and the potential is increasing from 0 to 1. Finally Theorems 2.4 and 2.5 are completely proved.

Remark that if $(\mathcal{H})$ is not fulfilled, then, the reduced model *has not a unique solution*. There is a finite number of solutions, the increasing potential φ_{CL} studied in the previous section, case *(iv)*, and the solutions constructed as above on the basis of φ_{CL}. It seems to be a difficult problem to chose between these multiple solutions. At this stage we can only say that the non monotone solutions ($p \geq 1$) seem to be less physical, but a stability study appears to be necessary in order to make this argument rigorous. This problem is linked to the non-uniqueness of the solution of the nonlinear system (2.12)-(2.20).

5. Conclusion

In this paper, we investigated the Child-Langmuir asymptotics applied to a bipolar plane diode. Let us come back to the dimensional variables (J_e, J_i, Φ). The anode is able to emit a current of ions $J_i^G < 0$ and the cathode is able to emit a current of electrons $J_e^G < 0$. Recall that $J_e^G = -|\overline{J}_e| j_e^g$ and $J_i^G = +|\overline{J}_i| j_i^g$.

We proved the existence of bipolar Child-Langmuir currents, which are different for the two species of particle since their mass are not identical. Let $J_{1\alpha}^{\mathrm{CL}}$, for $\alpha = e, i$ be the one-species Child-Langmuir currents corresponding to the electrons and the ions. The bipolar currents write

$$\begin{cases} J_{2e}^{\mathrm{CL}} = \varepsilon_0 \sqrt{\dfrac{2e}{m_e}} \dfrac{\Phi_L^{3/2}}{L^2}\, j_2^{\mathrm{CL}} \simeq 1.865 J_{1e}^{\mathrm{CL}}, \\ J_{2i}^{\mathrm{CL}} = \varepsilon_0 \sqrt{\dfrac{2e}{m_i}} \dfrac{\Phi_L^{3/2}}{L^2}\, j_2^{\mathrm{CL}} \simeq 1.865 J_{1i}^{\mathrm{CL}}. \end{cases}$$

If $|J_i^G| < 9J_{2i}^{\mathrm{CL}}$ or $|J_e^G| < 9J_{2e}^{\mathrm{CL}}$ then we showed that the effective currents flowing through the diode cannot exceed these bipolar Child-Langmuir currents. If both emissions are space-charge limited then the electric field vanishes on each electrode. This behaviour and these values are in accordance with the physical intuition and experimentation [15]. Remark that

$$J_{2i}^{\mathrm{CL}}/J_{2e}^{\mathrm{CL}} = \sqrt{m_e/m_i} \ll 1,$$

so the typical current of ions flowing inside the diode will be much smaller than the electrons' one; it is well-known indeed that in an ion diode the secondary electrons emitted by the cathode will transport the main part of the energy provided.

If both injection currents exceed the values $9J_{2\alpha}^{\mathrm{CL}}$, then we cannot characterize uniquely the limit solution, and it may appear virtual electrodes inside the diode ([14], p.108): virtual cathodes at the points where the potential vanishes, and virtual anodes at the points where the potential takes the value Φ_L.

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References

[1] F. Alabau, K. Hamdache, Y. J. Peng, Analyse asymptotique du système de Vlasov-Poisson instationnaire dans le cas d'une diode plane, preprint n^o 95001 du Lab. de Math. Appl. de Bordeaux I.

[2] N. Ben Abdallah, *The Child-Langmuir regime for electron transport in a plasma including a background of positive ions*, Math. Models Methods Appl. Sci., **4**, 1994, pp. 409-438.

[3] N. Ben Abdallah, P. Degond, *The Child-Langmuir law for the Boltzmann equation of semiconductors*, SIAM J. Math. Anal., **26**, 1995, pp. 364-398.

[4] N. Ben Abdallah, *Convergence of the Child-Langmuir asymptotics of the Boltzmann equation of semiconductors*, SIAM J. Math. Anal., **27**, 1996, pp. 92-109.

[5] N. Ben Abdallah, P. Degond, P. Markowich, *The quantum Child-Langmuir problem*, to appear.

[6] N. Ben Abdallah, P. Degond, F. Méhats, *Mathematical models for magnetic insulation*, to appear.

[7] N. Ben Abdallah, P. Degond, F. Méhats, *The Child-Langmuir asymptotics applied to magnetized flows*, to appear.

[8] P. Degond, *The Child-Langmuir Law in the Kinetic Theory of Charged Particles. Part 1, Electron Flows in Vacuum*, Advances in Kinetic Theory, B. Perthame Editor, World Scientific, Singapore, 1994.

[9] P. Degond, S. Jaffard, F. Poupaud, P.-A. Raviart, *The Child-Langmuir Asymptotics of the Vlasov-Poisson Equation for Cylindrically or Spherically Symmetric Diodes. Part 1 and Part 2*, Math. Methods Appl. Sci., **19**, 1996, pp. 287-312 and 313-340.

[10] P. Degond, P.-A. Raviart, *An asymptotic analysis for the one-dimensional Vlasov-Poisson system*, Asymptotic Anal., **4**, 1991, pp. 187-214.

[11] P. Degond, P.-A. Raviart, *On a penalization of the Child-Langmuir emission condition for the one-dimensional Vlasov-Poisson equation*, Asymptotic Anal., **6**, 1992, pp. 1-27.

[12] C. Greengard, P.-A. Raviart, *A Boundary Value Problem for the Stationary Vlasov-Poisson Equations: the Plane Diode*, Comm. Pure Appl. Math.,**43**, 1990, pp. 473-507, 1990.

[13] I. Langmuir, K. T. Compton, *Electrical Discharges in Gases: Part II. Fundamental phenomena in electrical discharges*, rev. Mod. Phys., **3**, 1931, pp. 191-257.

[14] J. D. Lawson, *The Physics of Charged Particle Beams*, Clarendon Press, Oxford, 1988.

[15] R. B. Miller, *An Introduction to the Physics of Intense Charged Particle Beams*, Plenum Press, New-York, 1982.

[16] J. W. Poukey, *Ion effects in relativistic diodes*, Appl. Phys. Let., **26**, n^0 4, 1975, pp. 145-146.

[17] F. Poupaud, *Boundary value problems for the stationary Vlasov-Maxwell system*, Forum Mathematicum, **4**, 1992, pp. 499-527.