The Child-Langmuir asymptotics for magnetized flows[1]

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Abstract

The Child langmuir asymptotics of the so called 1.5 dimensional Vlasov-Maxwell, modelling cross field diodes, is investigated. A set of limit problems including magnetically insulated and non insulated regimes are exhibited and analyzed. The solutions of the Vlasov-Maxwell system are shown to converge towards a solution of the limiting problem.

1. Introduction and singular perturbation problem

Electron transport in high energy devices such as vacuum diodes exhibits many nonlinear phenomena due to the extremely high applied voltages. One of these effects is the saturation of the current due to the self-consistent electric and magnetic fields. Langmuir and Compton [8] have investigated this phenomenon the first and established explicit formulae for the saturation current in the plane diode case, and approximate ones in the cylindrical and spherical diode cases. They assumed that the current saturates at a maximal value determined by the condition that the electric field vanishes at the emission cathode. This condition is referred to as the Child-Langmuir condition and the diode is said to operate under a space charge limited or a Child-Langmuir regime.

Later, Degond and Raviart [7] derived the Child-Langmuir law for plane diodes from a singular perturbation analysis of the Vlasov-Poisson system. They showed that the current flowing through the device tends to the Child-Langmuir current as a certain parameter that we shall not specify here tends to zero. This analysis gave rise to various extensions. First, the cylindrical and spherical diodes have been studied by Degond *et al.* [6] thanks to the supersolution concept developed by Poupaud [11]. The results of Langmuir and Compton have been recovered but the mathematical analysis of [6] showed that in some cases, the Child-Langmuir current is different from the maximal allowed value.

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Another difficulty appeared in the study by Ben Abdallah [1] of a plane diode containing a neutralizing ion background. The perturbation procedure, referred to as the Child-Langmuir asymptotics, possibly leads to multiple solutions due to the existence of closed trajectories which may be followed by trapped particles. The same non uniqueness problems have been encountered in the semiconductor case by Ben Abdallah and Degond [3, 2] and in the bipolar diode case by Méhats [10].

In the present work, we shall investigate the Child-Langmuir asymptotics for magnetized flows. Electrons are submitted to both electrostatic and magnetostatic forces. The model that we shall consider is the 1.5 dimensional stationary relativistic Vlasov-Maxwell system. In addition to the multiplicity of solutions of the limit problem, the analysis involves another difficulty due to the possible existence of a free boundary.

In this introduction, we summarize the physical setting which is detailed in [4]. In this paper, we rather focus on the mathematical analysis. Electrons are assumed to be confined in a plane denoted by xOy and submitted to an electrostatic field depending on and directed towards the x direction and a magnetic field depending on the x variable and having the z direction (orthogonal to x and y). The flow is assumed to be stationary so that the electron distribution function f is a solution of the stationary Vlasov equation in which the electromagnetic field derives from the electric and magnetic potentials (φ, a) which are computed self-consistently. The flow is relativistic so that the particle velocity (v_x, v_y) and the kinetic energy ϵ are given in terms of the particle momentum $\mathbf{p} = (p_x, p_y)$ by the dimensionless relativistic formulae

$$\mathbf{v} = (v_x, v_y) = \frac{\mathbf{p}}{\sqrt{1+\mathbf{p}^2}} \quad , \quad \epsilon = \sqrt{1+\mathbf{p}^2} - 1. \tag{1.1}$$

The singular perturbation problem associated to the stationary Vlasov-Maxwell system is written in scaled form (see [4] for a justification of the scaling):

$$\left\{ \begin{array}{ll} v_x \dfrac{\partial f^\varepsilon}{\partial x} + \left(\dfrac{d\varphi^\varepsilon}{dx} - v_y \dfrac{da^\varepsilon}{dx} \right) \dfrac{\partial f^\varepsilon}{\partial p_x} + v_x \dfrac{da^\varepsilon}{dx} \dfrac{\partial f^\varepsilon}{\partial p_y} = 0, & (1.2) \\ \qquad\qquad\qquad\qquad (x, p_x, p_y) \in (0,1) \times \mathbb{R}^2, & \\ \dfrac{d^2 \varphi^\varepsilon}{dx^2} = n^\varepsilon(x), \quad x \in (0,1), & (1.3) \\ \dfrac{d^2 a^\varepsilon}{dx^2} = j_y^\varepsilon(x), \quad x \in (0,1), & (1.4) \end{array} \right.$$

$$\left\{ \begin{array}{rcll} n^\varepsilon &=& \displaystyle\int_{\mathbb{R}^2} f^\varepsilon(p_x, p_y)\; dp_x dp_y, & (1.5) \\ j_y^\varepsilon &=& \displaystyle\int_{\mathbb{R}^2} v_y f^\varepsilon(p_x, p_y)\; dp_x dp_y & \\ &=& \displaystyle\int_{\mathbb{R}^2} \frac{p_y}{\sqrt{1+|\mathbf{p}|^2}} f^\varepsilon(p_x, p_y)\; dp_x dp_y, & (1.6) \end{array} \right.$$

$$
\begin{cases}
f^\varepsilon(0,p_x,p_y) = g^\varepsilon(p_x,p_y) = \dfrac{1}{\varepsilon^3}\, g\left(\dfrac{p_x}{\varepsilon},\dfrac{p_y}{\varepsilon}\right), & p_x > 0, \qquad (1.7)\\
f^\varepsilon(1,p_x,p_y) = 0, & p_x < 0, \qquad (1.8)\\
\varphi^\varepsilon(0) = 0, \quad \varphi^\varepsilon(1) = \varphi_L, & \qquad (1.9)\\
a^\varepsilon(0) = 0, \quad a^\varepsilon(1) = a_L. & \qquad (1.10)
\end{cases}
$$

where φ_L and a_L are the dimensionless versions of the known applied potentials. This paper is concerned with the limit $\varepsilon \to 0$ (child-Langmuir asymptotics).

Two quantities will be important in our analysis, namely

- the energy $\mathcal{W}(x,\mathbf{p}) = \epsilon(\mathbf{p}) - \varphi(x)$, (1.11)
- the canonical momentum $\mathcal{P}_y(x,\mathbf{p}) = p_y - a(x)$. (1.12)

which are invariants of the electron motion. Using the supersolution technique of Poupaud [11] based on these invariants, it is not difficult to obtain the following existence theorem

Theorem 1.1 *Let $\varepsilon > 0$ be fixed and g be an entering data that satisfies*

$$0 \le g^\varepsilon(p_x,p_y) \le C_{0,\varepsilon}(1+|\mathbf{p}|^2)^{-\beta}, \quad C_{0,\varepsilon} > 0, \quad \beta > 1.$$

Then there exists at least one solution $(\mathcal{S}_\varepsilon) = (f^\varepsilon, \varphi^\varepsilon, a^\varepsilon)$ of the problem (1.2) - (1.10) that satisfies

$$f^\varepsilon(x,p_x,p_y) \le C_\varepsilon(1+|\mathbf{p}|^2)^{-\beta}, \quad C_\varepsilon > 0, \tag{1.13}$$

and

$$(\varphi^\varepsilon, a^\varepsilon) \in \left(W^{2,\infty}(0,1)\right)^2.$$

Moreover, if G is a C^1 function such that

$$0 \le g^\varepsilon(p_x,p_y) \le \frac{1}{\varepsilon^3} G\left(\frac{\epsilon(p_x,p_y)}{\varepsilon^2}, \frac{p_y}{\varepsilon}\right),$$

then this solution $(\mathcal{S}_\varepsilon)$ satisfies

$$0 \le f^\varepsilon(x,p_x,p_y) \le \frac{1}{\varepsilon^3} G\left(\frac{\epsilon(p_x,p_y) - \varphi^\varepsilon(x)}{\varepsilon^2}, \frac{p_y - a^\varepsilon(x)}{\varepsilon}\right). \tag{1.14}$$

The above constructed solution satisfies the following conservation properties:

Lemma 1.2 *Let* $(\mathcal{S}_\varepsilon) = (f^\varepsilon, \varphi^\varepsilon, a^\varepsilon)$ *be the solution of* (1.2)-(1.10) *given by Theorem 1.1. We have*

$$\frac{d}{dx} j_x^\varepsilon = 0, \quad \frac{d}{dx} h^\varepsilon = 0,$$

where j_x^ε *and* h^ε *are the* x *component of the electronic current and the flux of momentum in the* x*-direction and are defined by*

$$j_x^\varepsilon(x) = \int_{\mathbb{R}^2} v_x f^\varepsilon(p_x, p_y)\ dp_x dp_y = \int_{\mathbb{R}^2} \frac{p_x}{\sqrt{1+|\mathbf{p}|^2}} f^\varepsilon(p_x, p_y)\ dp_x dp_y, \tag{1.15}$$

$$h^\varepsilon(x) = \int_{\mathbb{R}^2} v_x p_x f^\varepsilon(x, p_x, p_y) d\mathbf{p}\ - \frac{1}{2}\left[\left(\frac{d\varphi^\varepsilon}{dx}\right)^2 - \left(\frac{da^\varepsilon}{dx}\right)^2\right]. \tag{1.16}$$

By abuse of language, we shall call h^ε the "total energy". The injected current is finally denoted by

$$j_x^g = \int_{\mathbb{R}^2_+} f^\varepsilon(0, \mathbf{p}) d\mathbf{p} = \int_{\mathbb{R}^2_+} g(\mathbf{p}) d\mathbf{p},$$

where $\mathbb{R}^2_+ = \{\mathbf{p} \in \mathbb{R}^2 : p_x > 0\}$.

The outline of the paper is as follows. In Section 2, we shall state the main results, namely we give the set of problems to which the limit $\varepsilon \to 0$ leads. In Section 3, we prove that each of these problems is uniquely solvable and exhibit their solutions quasi explicitly. Finally the convergence proofs are deferred to Section 4.

2. Statement of the main results

The following function is called the *effective potential*

$$\theta(x) = (1+\varphi(x))^2 - a(x)^2 - 1. \tag{2.1}$$

We denote by $\theta_L = \theta(1) = (1+\varphi_L)^2 - a_L^2 - 1$. The qualitative behavior of the diode mainly exhibits two regimes depending on the sign of θ_L. When $\theta_L > 0$, the diode is *non magnetically insulated* whereas when $\theta_L < 0$, the diode is *magnetically insulated.* The threshold $\theta_L = 0$ can be expressed in terms of the so-called Hull cut-off magnetic field

$$a_L^H = \sqrt{\varphi_L^2 + 2\varphi_L}.$$

The diode is magnetically insulated if $a_L > a_L^H$. The limiting case $\theta_L = 0$ will not be considered in this paper.

2.1. Weak magnetic field: the non magnetically insulated diode ($\theta_L > 0$)

Later on, we prove that the limit model as $\varepsilon \to 0$ is characterized by the following properties:

(1.i) $\varphi(0) = 0, \quad a(0) = 0, \quad \varphi \geq 0$ and $a \geq 0$ on $[0,1]$;

(1.ii) $\varphi(1) = \varphi_L, \quad a(1) = a_L$;

(1.iii) $\theta \geq 0$ on $[0,1]$;

(1.iv) $\exists p \in \mathbb{N} \quad \exists (x_0, \ldots, x_p, x_{p+1}) \in [0,1]^{p+2}$ such that $0 = x_0 < x_1 < \ldots < x_p < x_{p+1} = 1$ and $\{x : \theta(x) = 0\} = \{x_0, \ldots, x_p\}$;

(1.v) $\exists (\bar{\jmath}_0, \bar{\jmath}_1, \ldots, \bar{\jmath}_p) \in (\mathbb{R}_+^*)^{p+1}$ such that on (x_k, x_{k+1}) we have

$$\begin{cases} \varphi'' \sqrt{\theta} = \bar{\jmath}_k \, (1 + \varphi) \\ a'' \sqrt{\theta} = \bar{\jmath}_k \, a \end{cases}$$

(1.vi) $\bar{\jmath}_p = \min\limits_{0 \leq k \leq p} \{\bar{\jmath}_k\}$;

(1.vii) $\varphi'(0) = 0$.

We set

$$\mathcal{F}^{\mathbf{ni}} = \Big\{ (\varphi, a) \in W^{2,1}([0,1]) \text{ verifying } (1,i), (1,ii), (1,iii), (1,iv), (1,v), (1.vi) \Big\}$$

and

$$\mathcal{F}^{\mathbf{ni}}_{\mathrm{CL}} = \Big\{ (\varphi, a) \in \mathcal{F}^{\mathbf{ni}} \text{ verifying } (1,vii) \Big\}.$$

Theorem 2.1 *Assume $\theta_L > 0$ and let $(f^\varepsilon, \varphi^\varepsilon, a^\varepsilon, j_x^\varepsilon)$ be a solution of* (1.2)-(1.10) *given by Theorem* 1.1 *where the injection profile g is assumed to be compactly supported. Then, as $\varepsilon \to 0$ and after extraction of a subsequence, there holds*

$$\begin{cases} (\varphi^\varepsilon, a^\varepsilon) \to (\varphi, a) \in \mathcal{F}^{\mathbf{ni}} \ \textit{in } (W^{2,1}([0,1]))^2 \ \textit{strong}, \\ f^\varepsilon \to f \ \textit{in } \mathcal{M}_b([0,1] \times \mathbb{R}^2) \ \textit{weak } *, \\ j_x^\varepsilon \to j_x, \end{cases}$$

where

$$f(x,p) = n^+(x) \delta\left(p_x - \sqrt{\theta(x)}\right) \delta\left(p_y - a(x)\right) + n^-(x) \delta\left(p_x + \sqrt{\theta(x)}\right) \delta\left(p_y - a(x)\right) \tag{2.2}$$

$$\textit{and} \quad n^+(x) = \frac{(\bar{\jmath}_k + j_x)}{2} \frac{1 + \varphi(x)}{\sqrt{\theta(x)}}, \quad n^-(x) = \frac{(\bar{\jmath}_k - j_x)}{2} \frac{1 + \varphi(x)}{\sqrt{\theta(x)}} \quad \textit{on } (x_k, x_{k+1}).$$

Moreover, $j_x = \bar{\jmath}_p$ and there exists j_x^{max} such that

$$j_x^g \geq j_x^{max} \Longrightarrow (\varphi, a) \in \mathcal{F}^{\mathbf{ni}}_{CL}. \tag{2.3}$$

In this theorem, $\mathcal{M}_b([0,1]\times\mathbb{R}^2)$ denotes the space of bounded measures on $[0,1]\times\mathbb{R}^2$.

Theorem 2.2 (Characterization of $\mathcal{F}_{\mathrm{CL}}^{\mathbf{ni}}$)
Let F be the set of subsets of $[0,1]$ with finite number of of elements. The following mapping

$$\begin{array}{ccl} \mathcal{F}_{CL}^{\mathbf{ni}} & \longrightarrow & F \\ (\varphi,a) & \longrightarrow & \{x_0,x_1,\cdots,x_p\}=\{x\in[0,1], \quad s.t.\ \theta(x)=0\} \end{array}$$

is an injection.

Remark 2.3 *The solution (φ,a) corresponding to $p=0$ has been studied in [4], where it was denoted $\mathcal{S}^{\mathbf{ni}}$. Moreover we can prove the following claim:*

$$\text{if } \theta_L\ge 1 \text{ then the limit model admits only one solution, such that } p=0 \,:\, \mathcal{F}_{CL}^{\mathbf{ni}}=\left\{\mathcal{S}^{\mathbf{ni}}\right\}. \tag{2.4}$$

Remark 2.4 *The points $(x_0,...,x_p)$ where θ vanishes are the turning points of the electron trajectories. Consequently, when $p\ge 1$, closed trajectories and trapped particles may appear (see Figure 1). In the case $p=0$, the electronic flow is monokinetic and the unique element of $\mathcal{F}_{CL}^{\mathbf{ni}}$ is the solution of the following semilinear elliptic system*

$$\left\{\begin{array}{lcll} \varphi'' & = & \dfrac{j_x\,(1+\varphi)}{\sqrt{(1+\varphi)^2-a^2-1}}, & \varphi(0)=0,\quad \varphi(1)=\varphi_L \\ a'' & = & \dfrac{j_x\,a}{\sqrt{(1+\varphi)^2-a^2-1}}, & a(0)=0,\quad a(1)=a_L \end{array}\right.$$

with the constraint $\varphi'(0)=0$.

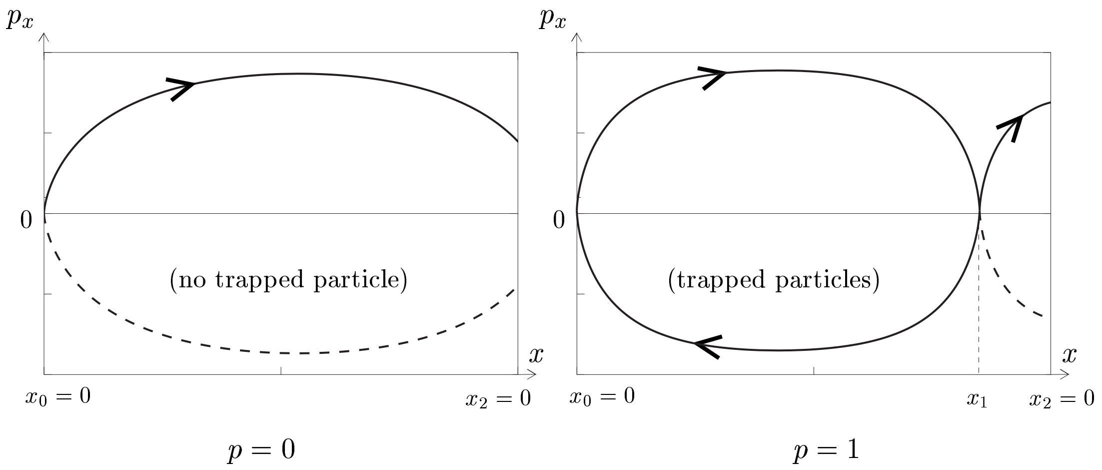


Figure 1 : *Two phase portraits in the case $\theta_L>0$*

Remark 2.5 *Condition* (2.3) *itself is the Child-Langmuir law : the system adjusts the value of the current in such a way that the electric field vanishes at the cathode (in usual situations the injected current is very high and condition* $j_x^g \geq j_x^{max}$ *is fulfilled).*

2.2. Strong magnetic field: the magnetically insulated diode ($\theta_L < 0$)

In this case the limit model is characterized by

(2.i) $\varphi(0) = 0, \quad a(0) = 0, \quad \varphi \geq 0$ and $a \geq 0$ on $[0, 1]$;

(2.ii) $\varphi(1) = \varphi_L, \quad a(1) = a_L$;

(2.iii) $\exists x^* \in (0, 1)$ such that $\theta \geq 0$ on $[0, x^*]$ and $\theta < 0$ on $(x^*, 1]$;

(2.iv) $a''(1 + \varphi) - a\varphi'' = 0$ on $[0, 1]$;

(2.v) For any interval (x_1, x_2) on which $\theta > 0$, there exists a constant $\bar{\jmath}_{(x_1,x_2)} > 0$ such that, on (x_1, x_2), $\begin{cases} \varphi'' \sqrt{\theta} = \bar{\jmath}_{(x_1,x_2)} (1 + \varphi) \\ a'' \sqrt{\theta} = \bar{\jmath}_{(x_1,x_2)} \, a \end{cases}$

(2.vi) $\varphi'' \equiv a'' \equiv 0$ on $(x^*, 1]$;

(2.vii) $\varphi'(0) = 0$.

We set

$$\mathcal{F}^{\mathbf{i}} = \Big\{(\varphi, a) \in C^1([0, 1]) \text{ verifying } (1, i), (1, ii), (1, iii), (1, iv), (1, v), (1, vi)\Big\}$$

and

$$\mathcal{F}^{\mathbf{i}}_{\mathrm{CL}} = \Big\{(\varphi, a) \in \mathcal{F}^{\mathbf{i}} \text{ verifying } (1, vii)\Big\} .$$

Theorem 2.3 *Assume* $\theta_L < 0$ *and let* $(f^\varepsilon, \varphi^\varepsilon, a^\varepsilon)$ *be a solution of* (1.2)-(1.10) *given by Theorem* 1.1. *Then, as* $\varepsilon \to 0$ *and after extraction of a subsequence, there holds*

$$\begin{cases} (\varphi^\varepsilon, a^\varepsilon) \to (\varphi, a) \in \mathcal{F}^{\mathbf{i}} \text{ in } (C^1([0, 1]))^2 \text{ strong,} \\ f^\varepsilon \to f = \text{ in } \mathcal{M}_b([0, 1] \times \mathbb{R}) \text{ weak } *. \end{cases}$$

Moreover there exists j_x^{max} *such that*

$$j_x^g \geq j_x^{max} \Longrightarrow (\varphi, a) \in \mathcal{F}^{\mathbf{i}}_{CL}. \tag{2.5}$$

Theorem 2.4 (Characterization of $\mathcal{F}^{\mathbf{i}}_{\mathrm{CL}}$)
Let $\mathcal{O}(0,1)$ be the set of open subsets of $(0,1)$. The mapping

$$\begin{array}{rl} \mathcal{F}^{\mathbf{i}} & \longrightarrow \mathcal{O}(0,1) \\ (\varphi,a) & \longrightarrow \{t\in(0,1) : \theta(tx^*)>0\} \end{array}$$

is one to one and onto.

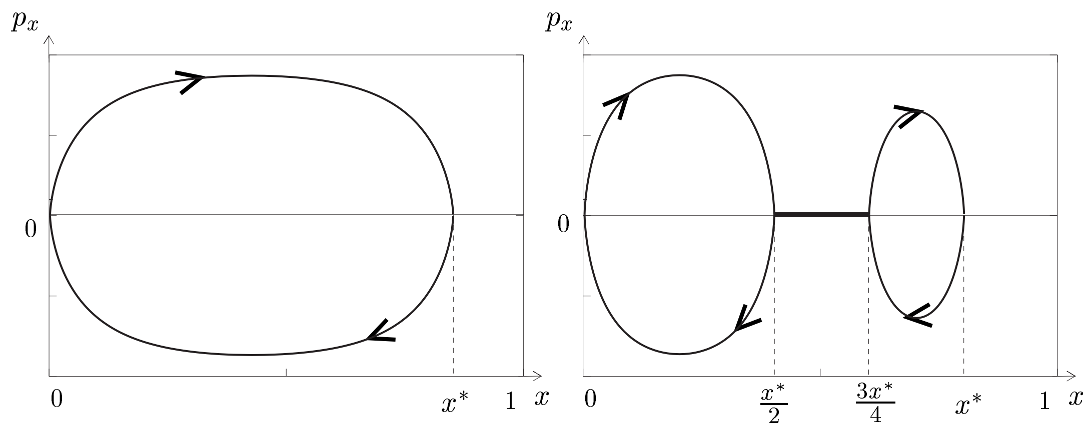
Remark 2.8 *Let $\Omega=\{x\in(0,x^*),\ s.t.\ \theta(x)>0\}$. Two limit cases are usually considered in the physical literature [5]. These cases have been studied in details in [4]:*
- the quasi laminar case, denoted by $\mathcal{S}^{\mathbf{i}}_{QL}$, corresponds to $\Omega=(0,x^*)$;
- the laminar case, also called parapotential or Brillouin flow, denoted by $\mathcal{S}^{\mathbf{i}}_{L}$, corresponds to $\Omega=\emptyset$.

0

Remark 2.9 *Since the total x current j_x vanishes in the limit $\varepsilon\to 0$, the electron densities n^+ and n^- on $(0,x^*)$ appearing in (2.2) (which still holds in the insulated case) are equal. We then have $n=2n^+=2n^-$ and*

$$f=\frac{n(x)}{2}\delta\left(p_x-\sqrt{\theta(x)}\right)\delta\left(p_y-a(x)\right)+\frac{n(x)}{2}\delta\left(p_x+\sqrt{\theta(x)}\right)\delta\left(p_y-a(x)\right), \quad (2.6)$$

On Figure 2, we have depicted the phase space in two cases: first, the quasilaminar case, second a more general case (referred to as "parasitic" in [4]) where trapped particles can occur. On the figure, we have chosen $\Omega=(0,\frac{x^*}{2})\cup(\frac{3x^*}{4},x^*)$.



Quasi laminar case *Parasitic case*

Figure 2 : Two phase portraits in the case $\theta_L<0$

3. Resolution of the limit models

In this section we prove the existence of solutions for the reduced models $\mathcal{F}_{\mathrm{CL}}^{\mathbf{ni}}$ and $\mathcal{F}_{\mathrm{CL}}^{\mathbf{i}}$ and prove Theorems 2.2 and 2.4. We first analyze the structure of the equations in the sheath $(0, x^*)$, and then turn to the resolution of the models on the whole domain (x^* is equal to 1 in the noninsulated case). The approach relies on a shooting method, where $\beta = a'(0)$, the magnetic field at the cathode, is the shooting parameter.

3.1. Preliminary study of the equations

Consider the equations of $\mathcal{F}_{\mathrm{CL}}^{\mathbf{ni}}$ and $\mathcal{F}_{\mathrm{CL}}^{\mathbf{i}}$ with the condition

$$a'(0) = \beta \tag{3.1}$$

instead of (1.*ii*) and (2.*ii*) and set $x^* = 1$ in the case of the noninsulated diode so that Conditions (1.*iii*) and (2.*iii*) read in both cases

$$\theta(x) \geq 0 \quad \text{for } x \in [0, x^*]. \tag{3.2}$$

Following [9] or [12], we shall solve the problem on $[0, x^*]$ in terms of the following functions

$$\zeta(x) = \sqrt{\theta(x)} = \sqrt{(1+\varphi)^2(x) - a^2(x) - 1}, \tag{3.3}$$

and

$$\gamma(x) = \mathrm{Argtanh}\left(\frac{a(x)}{1+\varphi(x)}\right) \tag{3.4}$$

where, for $t \in (-1, +1)$, $\mathrm{Argtanh}(t)$ is the unique real number u such that $\tanh(u) = \sinh(u)/\cosh(u) = t$. The original functions φ and a can be computed in terms of ζ and γ by

$$\left\{\begin{array}{l} 1+\varphi(x) = \sqrt{\theta(x)+1}\cosh(\gamma(x)), \\ a(x) = \sqrt{\theta(x)+1}\sinh(\gamma(x)). \end{array}\right. \tag{3.5}$$

To obtain a set of equations satisfied by ζ and γ, we first integrate (2.*iv*) (which holds also in the noninsulated case) and obtain

$$(1+\varphi)a' - \varphi' a = \beta. \tag{3.6}$$

We deduce from this equation and (3.5) that, for $x \in [0, x^*]$,

$$\gamma'(x) = \frac{\beta}{\theta(x)+1} = \frac{\beta}{\zeta^2(x)+1}. \tag{3.7}$$

This equation completely determines $\gamma(x)$ as soon as $\zeta(x)$ is known. We now concentrate on the determination of ζ on the largest interval $I = (x_-, x_+)$ such that $\theta(x_-) = 0$ and $\theta > 0$ on I. We first notice that

$$\theta'(x_-) = 0. \tag{3.8}$$

Indeed, either $x_- > 0$, and the above condition is a direct consequence of the positivity and the C^1 regularity of θ on $[0, x^*]$ or $x_- = 0$ and since $\varphi(0) = \varphi'(0) = a(0) = 0$, we have

$$\theta'(0) = 2(1 + \varphi(0))\varphi'(0) - 2a(0)a'(0) = 2\varphi'(0) = 0.$$

Let $\gamma_- = \gamma(x_-)$. The differentiation of (3.5) at $x = x_-$ leads, in view of (3.7) and (3.8), to

$$\begin{cases} 1 + \varphi(x_-) = \cosh(\gamma_-), \quad a(x_-) = \sinh(\gamma_-), \\ \varphi'(x_-) = \beta \sinh(\gamma_-), \quad a'(x_-) = \beta \cosh(\gamma_-). \end{cases} \tag{3.9}$$

Moreover by (1.v) or (2.v) there exists a constant $\bar{\jmath} > 0$ such that, on I,

$$\begin{cases} \varphi'' \zeta = \bar{\jmath}\,(1 + \varphi) \\ a'' \zeta = \bar{\jmath}\, a. \end{cases} \tag{3.10}$$

Multiplying the first equation of (3.10) by $2\varphi'$ and the second one by $2a'$, we obtain

$$(2\varphi'\varphi'' - 2a'a'')\zeta = 2\bar{\jmath}\left((1 + \varphi)\varphi' - aa'\right);$$

which yields (since $\zeta \neq 0$ on I)

$$\left((\varphi')^2 - (a')^2\right)' = 2\bar{\jmath}\zeta'.$$

Integrating this equation gives in view of (3.9),

$$2\bar{\jmath}\zeta - (\varphi')^2 + (a')^2 = \beta^2. \tag{3.11}$$

Substituting in this equation φ' and a' by their expression coming from the differentiation of (3.5) gives the following differential equation on ζ

$$(\zeta')^2 = \frac{2\bar{\jmath}}{\zeta}\left(\zeta^2 - \frac{\beta^2}{2\bar{\jmath}}\zeta + 1\right). \tag{3.12}$$

Several cases appear depending on the fact that the right hand side of this equation vanishes or not. Set $\tilde{\beta}^2 = \dfrac{\beta^2}{\bar{\jmath}}$.

(**a**) $\tilde{\beta} < 2$; the right hand side cannot vanish. Hence the sign of ζ' does not change and, since $\zeta \geq 0$ and $\zeta(x_-) = 0$, ζ' remains always positive. Equation (3.12) can thus be integrated on $[x_-, +\infty)$ leading to the implicit expression of ζ

$$x = x_- + \frac{1}{\sqrt{\bar{\jmath}}}\int_0^{\zeta(x)} \frac{\sqrt{\omega}}{\sqrt{2(\omega^2 - \frac{\tilde{\beta}^2}{2}\omega + 1)}}d\omega. \tag{3.13}$$

Finally $\gamma(x)$ can be deduced from (3.7) and (3.13):

$$\gamma(x) = \gamma_- + \int_0^{\zeta(x)} \frac{\tilde{\beta}\sqrt{\omega}}{\sqrt{2(\omega^2 - \frac{\tilde{\beta}^2}{2}\omega + 1)}(\omega^2 + 1)}d\omega. \tag{3.14}$$

(b) $\tilde{\beta} > 2$; the right-hand side vanishes when ζ takes the value

$$\zeta_m = \frac{\tilde{\beta}^2 - \sqrt{\tilde{\beta}^4 - 16}}{4}, \tag{3.15}$$

and is positive for $\zeta < \zeta_m$. We set

$$x_m = x_- + \frac{1}{\sqrt{\jmath}} \int_0^{\zeta_m} \frac{\sqrt{\omega}}{\sqrt{2(\omega^2 - \frac{\tilde{\beta}^2}{2}\omega + 1)}} d\omega. \tag{3.16}$$

Since ζ_m is a simple root of the $\omega^2 - \frac{\tilde{\beta}^2}{2}\omega + 1$, the integral in (3.16) converges and therefore $x_m < +\infty$. For $x \in [x_-, x_m)$, one can solve (3.12) as in case (**a**), yielding the same expressions (3.13) and (3.14), as above. The differentiation of (3.12) gives

$$\zeta'' = -\frac{\jmath}{\zeta^2}\left(\zeta^2 - \frac{\tilde{\beta}^2}{2}\zeta + 1\right) + \frac{\jmath}{\zeta}\left(2\zeta - \frac{\tilde{\beta}^2}{2}\right).$$

Hence $\zeta''(x_m) = \frac{\jmath}{\zeta_m}\left(2\zeta_m - \frac{\tilde{\beta}^2}{2}\right) < 0$ since $\tilde{\beta} > 2$ and ζ_m is the smallest root of (3.12). Thus ζ' is strictly decreasing in a neighbourhood of x_m. It is then clear from (3.12) that ζ is symmetric with respect to x_m. Therefore, ζ is positive on (x_-, x_+) and vanishes at x_- and x_+ where $x_+ = 2x_m - x_-$

$$\zeta(x) = \zeta(2x_m - x), \qquad \text{for } x \in [x_m, 2x_m - x_-], \tag{3.17}$$

and we have

$$\gamma(x) = 2\gamma_m - \gamma(2x_m - x), \qquad \text{for } x \in [x_m, 2x_m - x_-], \tag{3.18}$$

where

$$\gamma_m = \gamma(x_m) = \gamma_- + \int_0^{\zeta_m} \frac{\tilde{\beta}\sqrt{\omega}}{\sqrt{2(\omega^2 - \frac{\tilde{\beta}^2}{2}\omega + 1)}(\omega^2 + 1)} d\omega.$$

(c) $\tilde{\beta} = 2$; the right hand side admits a double root $\zeta_m = 1$, and one can again define $\zeta(x)$ implicitly by (3.13), for every $x \geq x_-$. Indeed, the critical point ζ_m is only reached for $x \to +\infty$. The function $\gamma(x)$ is defined by (3.14). Moreover, we have $\zeta(x) \to \zeta_m$ and $\gamma(x) \to +\infty$ as $x \to +\infty$.

To rewrite these results in a more compact way, we introduce the following auxiliary functions:

$$\Lambda_a(\tilde{\beta}, \zeta) = \int_0^{\zeta} \frac{\tilde{\beta}\sqrt{\omega}}{\sqrt{2(\omega^2 - \frac{\tilde{\beta}^2}{2}\omega + 1)}(\omega^2 + 1)} d\omega, \tag{3.19}$$

$$\Pi_a(\tilde{\beta}, \zeta) = \int_0^{\zeta} \frac{\tilde{\beta}\sqrt{\omega}}{\sqrt{2(\omega^2 - \frac{\tilde{\beta}^2}{2}\omega + 1)}} d\omega; \tag{3.20}$$

If $\tilde\beta \le 2$ these functions are defined for all $\zeta \ge 0$. If $\tilde\beta > 2$ they are only defined for $\zeta \in [0, \zeta_m]$, where the function $\zeta_m(\beta)$ is given by (3.15). In the case $\tilde\beta > 2$, we set

$$\Lambda_m(\tilde\beta) = \Lambda_a(\tilde\beta, \zeta_m(\tilde\beta)), \quad \Pi_m(\tilde\beta) = \Pi_a(\tilde\beta, \zeta_m(\tilde\beta)), \tag{3.21}$$

$$\Lambda_b(\tilde\beta, \zeta) = 2\Lambda_m(\tilde\beta) - \Lambda_a(\tilde\beta, \zeta), \quad \Pi_b(\tilde\beta, \zeta) = 2\Pi_m(\tilde\beta) - \Pi_a(\tilde\beta, \zeta). \tag{3.22}$$

Thanks to these functions, the solution of the Cauchy problem (3.9), (3.10) is written as follows:
• If $\tilde\beta \le 2$, the solution $(\zeta(x),\ \gamma(x))$ is defined on $[x_-, +\infty)$ and

$$\gamma(x) = \gamma_- + \Lambda_a(\tilde\beta, \zeta(x)), \quad \beta x = \beta x_- + \Pi_a(\tilde\beta, \zeta(x)). \tag{3.23}$$

• If $\tilde\beta > 2$, the solution $(\zeta(x), \gamma(x))$ is defined on $[x_-, 2x_m - x_-]$ where

$$\beta x_m = \beta x_- + \Pi_m(\tilde\beta) \tag{3.24}$$

$$\begin{cases} \text{If } x \in [x_-, x_m], & \gamma(x) = \gamma_- + \Lambda_a(\tilde\beta, \zeta(x)), \quad \beta x = \beta x_- + \Pi_a(\tilde\beta, \zeta(x)), \\ \text{If } x \in [x_m, 2x_m - x_-], & \gamma(x) = \gamma_- + \Lambda_b(\tilde\beta, \zeta(x)), \quad \beta x = \beta x_- + \Pi_b(\tilde\beta, \zeta(x)). \end{cases} \tag{3.25}$$

Remark that the current $\bar\jmath$ only appears in (3.23) and (3.25) through the parameters $\tilde\beta$ and β, thanks to the equality $\beta = \sqrt{\bar\jmath}\tilde\beta$.

3.2. Noninsulated case

Assume now $\theta_L > 0$. In this subsection we prove Theorem 2.2 in two steps. Here $p \in \mathbb{N}$ and $(x_0, x_1, ..., x_p)$ are fixed such that $0 = x_0 < x_1 < ... < x_p < 1$. We denote $x_{p+1} = 1$. Since $x^* = 1$, ζ and γ are defined on $[0, 1]$. Denote

$$\zeta_L = \sqrt{\theta_L} = \sqrt{(1+\varphi_L)^2 - 1 - a_L^2}, \quad \gamma_L = \operatorname{Argtanh}\left(\frac{a_L}{1+\varphi_L}\right).$$

The boundary condition $(1.ii)$ is thus equivalent to

$$\zeta(1) = \zeta_L, \quad \gamma(1) = \gamma_L, \tag{3.26}$$

the functions ζ and γ being the new unknowns of $\mathcal{F}^{\mathbf{ni}}_{\mathrm{CL}}$. The complete determination of these functions requires the knowledge of $p+2$ parameters, namely β and $\bar\jmath_0, \dots, \bar\jmath_p$. In the first step, we reduce the number of parameters to one.

• *Step 1: Reduction to a single shooting parameter*
It is clear from the definition of x_k's that the interval (x_k, x_{k+1}) is of the form (x_-, x_+) for $k < p$. The discussion of the previous subsection leads in view of (3.25) to

$$\beta(x_{k+1} - x_k) = \Pi_b(\tilde\beta_k, \zeta(x_{k+1}))$$

and since $\zeta(x_{k+1}) = 0$, we have

$$\beta(x_{k+1} - x_k) = 2\Pi_m(\tilde{\beta}_k) \quad , \quad \gamma(x_{k+1}) - \gamma(x_k) = 2\Lambda_m(\tilde{\beta}_k)$$

where $\tilde{\beta}_k = \beta/\sqrt{\jmath_k}$. For $k = p$, we have thanks to (3.25)

$$\beta(1 - x_p) = \Pi_b(\tilde{\beta}_p, \zeta_L) \quad \text{or} \quad \beta(1 - x_p) = \Pi_a(\tilde{\beta}_p, \zeta_L).$$

This system is uniquely solvable in $\tilde{\beta}_p$. Indeed, after straightforward but lengthy computations we obtain the following lemma

Lemma 3.1 *Let $\zeta \geq 0$ be fixed. The continuous functions $\zeta_m(.)$, $\Pi_a(., \zeta)$, $\Pi_b(., \zeta)$ and $\Pi_m(.)$, respectively defined by (3.15), (3.20), (3.22) and (3.21), verify the following properties:*
(a) ζ_m is decreasing and maps $[2, +\infty)$ onto $(0, 1]$;
(b) if $\zeta \geq 1$ then $\Pi_a(., \zeta)$ is increasing and maps $[0, 2)$ onto $[0, +\infty)$;
$\Pi_b(., \zeta)$ is not defined;
(c) if $\zeta < 1$ then $\Pi_a(., \zeta)$ is increasing and maps $[0, \zeta_m^{-1}(\zeta)]$ onto $[0, \Pi_m(\zeta_m^{-1}(\zeta))]$;
$\Pi_b(., \zeta)$ is decreasing and maps $(2, \zeta_m^{-1}(\zeta)]$ onto $[\Pi_m(\zeta_m^{-1}(\zeta)), +\infty)$;
(d) Π_m is decreasing and maps $(2, +\infty)$ onto $(0, +\infty)$.

We notice that (d) is a particular case of (c), since we have $\Pi_b(., 0) = 2\Pi_m(.)$ and $\zeta_m^{-1}(0) = +\infty$. An immediate consequence of this lemma is the following

Lemma 3.2 *Let $(y, \zeta) \in (\mathbb{R}_+)^2$. The following system in $\tilde{\beta}$*

$$\Pi_a(\tilde{\beta}, \zeta) = y \quad \textit{or} \quad \Pi_b(\tilde{\beta}, \zeta) = y.$$

has a unique solution $\tilde{\beta} = \Pi_{a,b}^{(-1)}(y, \zeta)$ defined by

$$- \textit{if } \zeta \geq 1 \textit{ then } \left\{ \begin{array}{l} \Pi_{a,b}^{(-1)}(y, \zeta) := (\Pi_a(., \zeta))^{-1}(y) \\ \left(\Lambda\Pi^{(-1)}\right)(y, \zeta) := \Lambda_a\left((\Pi_a(., \zeta))^{-1}(y), \zeta\right) \end{array} \right.$$

$$- \textit{if } \zeta < 1 \textit{ and } y \in [0, \Pi_m(\zeta_m^{-1}(\zeta))] \textit{ then}$$

$$\left\{ \begin{array}{l} \Pi_{a,b}^{(-1)}(y, \zeta) := (\Pi_a(., \zeta))^{-1}(y) \\ \left(\Lambda\Pi^{(-1)}\right)(y, \zeta) := \Lambda_a\left((\Pi_a(., \zeta))^{-1}(y), \zeta\right) \end{array} \right.$$

$$- \textit{if } \zeta < 1 \textit{ and } y \in [\Pi_m(\zeta_m^{-1}(\zeta)), +\infty) \textit{ then}$$

$$\left\{ \begin{array}{l} \Pi_{a,b}^{(-1)}(y, \zeta) := (\Pi_b(., \zeta))^{-1}(y) \\ \left(\Lambda\Pi^{(-1)}\right)(y, \zeta) := \Lambda_b\left((\Pi_b(., \zeta))^{-1}(y), \zeta\right). \end{array} \right.$$

Moreover we have the following monotonicity properties:

Lemma 3.3 *Let $\zeta \geq 0$ be given. The above defined functions $\Pi_{a,b}^{(-1)}(.,\zeta)$ and $\left(\Lambda\Pi^{(-1)}\right)(.,\zeta)$ are one-to-one and onto from $\mathbb{R}_+$ to $\mathbb{R}_+$. Moreover $\left(\Lambda\Pi^{(-1)}\right)(.,\zeta)$ is increasing.*

The above lemmas prove that, as soon as β and the x_k's are known, $\tilde{\beta}_k$'s are uniquely defined by

$$(\forall k < p) \quad \tilde{\beta}_k = \Pi_m^{-1}\left(\frac{\beta(x_{k+1}-x_k)}{2}\right) \tag{3.27}$$

$$(\forall k < p) \quad \gamma(x_{k+1}) - \gamma(x_k) = 2\Lambda_m \circ \Pi_m^{-1}\left(\frac{\beta(x_{k+1}-x_k)}{2}\right) \tag{3.28}$$

and for $k = p$

$$\tilde{\beta}_p = \Pi_{a,b}^{(-1)}\left(\beta(1-x_p),\zeta_L\right). \tag{3.29}$$

$$\gamma_L - \gamma(x_p) = \left(\Lambda\Pi^{(-1)}\right)\left(\beta(1-x_p),\zeta_L\right). \tag{3.30}$$

Now the only point left to show is that β can be uniquely determined.

• *Step 2: determination of β*
By adding up (3.30) and (3.28), we end up with the following equation for β

$$\gamma_L = F_1\left(\beta,\zeta_L\right), \tag{3.31}$$

where

$$F_1\left(\beta,\zeta_L\right) = \sum_{k=0}^{p-1} 2\Lambda_m \circ \Pi_m^{-1}\left(\frac{\beta(x_{k+1}-x_k)}{2}\right) + \left(\Lambda\Pi^{(-1)}\right)\left(\beta(1-x_p),\zeta_L\right).$$

Moreover thanks to Lemma 3.3 and since $2\Lambda_m \circ \Pi_m^{-1}(y/2) = \left(\Lambda\Pi^{(-1)}\right)(y,0)$, for all $\zeta \geq 0$ the function $F_1(.,\zeta)$ is continuous and increasing from $\mathbb{R}_+$ onto $\mathbb{R}_+$. Therefore (3.31) admits a unique solution

$$\beta = \left(F_1(.,\zeta_L)\right)^{-1}(\gamma_L). \tag{3.32}$$

β being uniquely determined, by (3.29) and (3.27) the currents are given by

$$\begin{aligned} \text{for } 0 \leq k < p, \qquad \bar{\jmath}_k &= \left(\frac{\beta}{\Pi_m^{-1}\left(\frac{\beta(x_{k+1}-x_k)}{2}\right)}\right)^2 \\ \text{and} \quad \bar{\jmath}_p &= \left(\frac{\beta}{\Pi_{a,b}^{(-1)}\left(\beta(1-x_p),\zeta_L\right)}\right)^2. \end{aligned} \tag{3.33}$$

Theorem 2.2 is proved. □

We said that the mapping of Theorem 2.2 is only an injection. Let us explain it now. Indeed, for a given $(x_0,...,x_p)$ such that $x_0 = 0$, the above constructed (φ, a)

verifies $(1.i)$, $(1.ii)$, $(1.iii)$, $(1,iv)$, $(1,v)$ and $(1,vii)$, but Condition $(2.vi)$ has to be checked *a posteriori* : some distributions $(x_0, x_1, \ldots, x_p)$ may not provide a suitable solution. In particular assume that $\theta_L \geq 1$. Hence $\zeta_L \geq 1$. Lemma 3.1 (b), eq. (3.29) and the definition of $\Pi_{a,b}^{(-1)}$ imply

$$\tilde{\beta}_p = \Pi_a^{-1}(\beta(1-x_p), \zeta_L) < 2.$$

Moreover, if $p > 0$, by Lemma 3.1 (c) and eq. (3.27) we have

$$\forall k < p \quad \tilde{\beta}_k = (\Pi_b(.,0))^{-1}(\beta(x_{k+1} - x_k), \zeta_L) > 2.$$

Therefore, since $\tilde{\beta}_k = \beta/\sqrt{\bar{\jmath}_k}$, if $p > 0$ we have

$$\forall k < p \quad \bar{\jmath}_k < \bar{\jmath}_p,$$

and $(1.vi)$ cannot be fulfilled. From this, we can finally deduce (2.4).

3.3. Case of a strong magnetic field

In this subsection we assume $\theta_L < 0$ and prove Theorem 2.4. The set $\Omega_0 \in \mathcal{O}(0,1)$ is given and we have to show that there exists a unique solution of $\mathcal{F}_{\mathrm{CL}}^{\mathbf{i}}$ such that $\Omega_0 = \frac{1}{x^*}\Omega$, where $\Omega = \{x \in (0, x^*) : \theta(x) > 0\}$. The set Ω_0 can be written

$$\Omega_0 = \bigcup_{k \in \mathcal{I}} (\lambda_k, \mu_k),$$

where $\mathcal{I}$ is countable and for $i \neq j$, $(\lambda_i, \mu_i) \cap (\lambda_j, \mu_j) = \emptyset$. This decomposition is unique apart from a permutation of the indexes $k \in \mathcal{I}$. Remark that $\Omega = \{x \in (0, x^*) : \theta(x) > 0\}$ implies

$$\text{for } i \in \mathcal{I}, \quad \theta(\lambda_i x^*) = \theta(\mu_i x^*) = 0.$$

The parameters which define a solution belonging to $\mathcal{F}_{\mathrm{CL}}^{\mathbf{i}}$ are the edge x^* of the sheath and the constant real numbers $\bar{\jmath}_i$ associated with (λ_i, μ_i) by $(2.v)$. We proceed like in the proof of Theorem 2.2 and express the $\bar{\jmath}_i$'s in terms of the shooting parameter $\beta = a'(0)$.

- *Step 1: construction of the solution and determination of* $\bar{\jmath}_i$

In this step the pair (β, x^*) is given and the boundary conditions $\varphi(1) = \varphi_L$ and $a(1) = a_L$ are replaced by (3.1).

Lemma 3.4 *Let $(\beta, x^*) \in \mathbb{R}_+ \times (0,1)$. Then there exists a unique solution (φ, a) of $(2.i)$, $(2.iii)$, $(2.iv)$, $(2.v)$, $(2.vi)$, $(2.vii)$, (3.1). Moreover, if we denote by $|\Omega_0| = \sum_{i \in \mathcal{I}} (\mu_i - \lambda_i)$, we have*

$$\gamma(x^*) = \beta x^*(1 - |\Omega_0|) + \sum_{i \in \mathcal{I}} 2\Lambda_m \circ \Pi_m^{-1}\left(\frac{\beta x^*(\mu_i - \lambda_i)}{2}\right), \tag{3.34}$$

$$\begin{cases} 1+\varphi(x^*) = \cosh(\gamma(x^*)), & a(x^*) = \sinh(\gamma(x^*)), \\ \varphi'(x^*) = \beta\sinh(\gamma(x^*)), & a'(x^*) = \beta\cosh(\gamma(x^*)). \end{cases} \tag{3.35}$$

Proof. First, it is clear from the discussion of Subsection 3.1 that, since $(\lambda_i x^*, \mu_i x^*)$ is of the form (x_-, x_+), we have

$$\Pi_m\left(\frac{\beta}{\sqrt{\jmath_i}}\right) = \frac{\beta x^*(\mu_i - \lambda_i)}{2}.$$

By Lemma 3.1 the function $\Pi_m(y) \equiv \frac{1}{2}\Pi_b(y,0)$ is one-to-one and onto from $\mathbb{R}_+$ to $\mathbb{R}_+$ thus

$$\frac{\beta}{\sqrt{\jmath_i}} = \Pi_m^{-1}\left(\frac{\beta x^*(\mu_i - \lambda_i)}{2}\right). \tag{3.36}$$

This equation uniquely determines $\jmath_i$ and $\tilde{\beta}_i = \beta/\sqrt{\jmath_i}$ as soon as β and x^* are given. Consequently (3.25) uniquely defines ζ on Ω and since $\zeta \equiv 0$ on $[0, x^*]\backslash\Omega$, ζ is uniquely determined on $[0, x^*]$.

Let us now determine γ on $[0, x^*]$ (we recall that β and x^* are fixed). We first notice that

$$\gamma(\lambda_i x^*) - \gamma(\mu_i x^*) = 2\Lambda_m \circ \Pi_m^{-1}\left(\frac{\beta x^*(\mu_i - \lambda_i)}{2}\right).$$

Next we deduce from (3.7)

$$\gamma' \equiv \beta \quad \text{on } [0, x^*]\backslash\Omega.$$

Consequently, we have for every $x \in [0, x^*]\backslash\Omega$

$$\begin{aligned} \gamma(x) &= \int_0^x \gamma'(y)dy = \int_{(0,x)\backslash\Omega} \beta dy + \int_{(0,x)\cap\Omega} \gamma' dy \\ &= \beta\left(x - x^* \sum_{i\in\mathcal{I}(x)} (\mu_i - \lambda_i)\right) + \sum_{i\in\mathcal{I}(x)} 2\Lambda_m \circ \Pi_m^{-1}\left(\frac{\beta x^*(\mu_i - \lambda_i)}{2}\right) \end{aligned} \tag{3.37}$$

where

$$\mathcal{I}(x) = \{i \in \mathcal{I} : \lambda_i x^* \le x\}.$$

For $x \in (0, x^*) \cap \Omega$, $\gamma(x)$ is given by (3.25) where $\gamma_- = \gamma(\lambda_k x^*)$; in this expression, $k \in \mathcal{I}$ is the unique index such that $\lambda_k x^* < x < \mu_k x^*$ and $\gamma(\lambda_k x^*)$ is given by (3.37). Finally (3.34) is obtained by applying (3.37) to $x = x^*$.

To prove (3.35), we first notice that since $\theta \ge 0$ in the vicinity of $\lambda_i x^*$ and vanishes at $x = \lambda_i x^*$, we have $\theta'(\lambda_i x^*) = 0$. The symmetry of θ on $(\lambda_i x^*, \mu_i x^*)$ implies

$$\theta'(\mu_i x^*) = 0. \tag{3.38}$$

which insures, in view of the continuity of θ', that θ' vanishes on $\partial\Omega$. Consequently

$$\theta'(x^*) = 0. \tag{3.39}$$

Indeed, either $x^* \in \partial\Omega$ and the claim is already proven or $x^* \notin \partial\Omega$ and by definition θ vanishes on an interval of the form $[x^* - \alpha, x^*]$. Differentiating (3.5) at $x = x^*$ and using (3.7) and (3.39) leads immediately to (3.35).

The solution (φ, a) is now completely determined on $[0, x^*]$ (β and x^* being given). The C^1 extension of these function by affine functions outside the sheath (see(2.vi)) completes their determination on the whole interval $[0,1]$. $\square$

We now fit the parameters β and x^* so that (2.ii) holds:

- *Step 2: resolution on* $[x^*, 1]$

We first obtain some additional information on the various parameters of $\mathcal{F}^{\mathbf{i}}_{\mathrm{CL}}$. Denoting $\gamma^* = \gamma(x^*)$, we have

Lemma 3.5 *Let* (φ, a) *be solution of* (2.vi), (3.35). *Then the boundary conditions* (2.ii) *are fulfilled if and only if*

$$\gamma^* = \mathrm{Log}\frac{1 + \varphi_L + a_L}{1 + \sqrt{-\theta_L}} \tag{3.40}$$

and

$$\beta(1 - x^*) = \sqrt{-\theta_L}. \tag{3.41}$$

Proof. We only prove the "only if" statement. The "if" statement is easily checked afterwards. Let (φ, a) be a solution of (2.vi), (3.35). The potentials are affine functions outside the sheath. Then, (2.ii) yields

$$\varphi'(x^*) = \frac{\varphi_L - \varphi(x^*)}{1 - x^*}, \quad a'(x^*) = \frac{a_L - a(x^*)}{1 - x^*},$$

which implies, with (3.35),

$$\left\{ \begin{array}{l} (1 - x^*)\beta \sinh(\gamma^*) = 1 + \varphi_L - \cosh(\gamma^*) \\ (1 - x^*)\beta \cosh(\gamma^*) = a_L - \sinh(\gamma^*). \end{array} \right. \tag{3.42}$$

Dividing the first equation by the second one yields

$$(1 + \varphi_L) \cosh(\gamma^*) - a_L \sinh(\gamma^*) = 1, \tag{3.43}$$

which can be solved explicitly for γ^*, writing that $X = \exp(\gamma^*)$ is a solution of

$$P(X) := (a_L - \varphi_L - 1)X^2 + 2X - (1 + a_L + \varphi_L) = 0. \tag{3.44}$$

This second degree equation has two solutions since $\theta_L < 0$. Moreover, the trinomial $P(X)$ takes negative values for $X = 0$ and $X = 1$ so we distinguish two cases. If

$a_L \geq 1+\varphi_L$ then (3.44) admits only one solution greater than 1, thus (3.43) has one positive solution, which is defined by (3.40).

if $a_L < 1+\varphi_L$, then both roots of (3.44) are greater than 1 (notice that, by (3.44), their sum is positive), and (3.43) admits two positive solutions given by

$$\gamma^*_\pm = \mathrm{Log}\frac{1+\varphi_L+a_L}{1\pm\sqrt{-\theta_L}}\ .$$

Since $\cosh(\gamma_L) = 1+\varphi_L$, an easy computation shows that

$$\gamma^*_+ < \gamma_L < \gamma^*_-.$$

From (3.35) we deduce that ${\varphi'}^* \geq 0$. Thus, we must have $\varphi^* \leq \varphi_L$ and $\cosh(\gamma^*_+)$ is the only acceptable solution of (3.43). Therefore, (3.40) holds in both cases. Finally, (3.41) can be deduced from (3.42) and (3.40). □

- *Step 3: the shooting method*

By Lemma 3.5, we seek all the pairs (β, x^*) such that the solution given by Lemma 3.4 verifies (3.40) and (3.41). Let

$$F_2(y) = y(1-|\Omega_0|) + \sum_{i\in\mathcal{I}} 2\Lambda_m \circ \Pi_m^{-1}\left(\frac{y(\mu_i-\lambda_i)}{2}\right).$$

By (3.34), the condition (3.40) is written

$$F_2(\beta x^*) = \mathrm{Log}\frac{1+\varphi_L+a_L}{1+\sqrt{-\theta_L}}. \tag{3.45}$$

Moreover, like for $F_1(.,\zeta)$ and thanks to Lemma 3.3 and to the properties of the function $\Lambda_m\circ\Pi_m^{-1}$, F_2 is continuous, increasing and maps $\mathbb{R}_+$ onto $\mathbb{R}_+$. Consequently Equation (3.45) admits exactly one solution βx^*. Additionally, by (3.41), $\beta(1-x^*)$ is also uniquely defined which implies that there exists only one pair (β, x^*) such that (3.34), (3.40) and (3.41) hold:

$$\begin{cases} \beta = \sqrt{-\theta_L} + F_2^{-1}\left(\mathrm{Log}\dfrac{1+\varphi_L+a_L}{1+\sqrt{-\theta_L}}\right) \\ x^* = 1 - \dfrac{\sqrt{-\theta_L}}{\beta}. \end{cases}$$

Lemmas 3.4 and 3.5 enable us to conclude the proof of Theorem 2.4. Remark that the solution (φ, a), like F_2, is independent of the choice of the decomposition in (λ_i, μ_i) of the open set Ω_0, since it does not depend on the order of the indices. □

4. Convergence to the limit models and limitation of the current

In this part we prove Theorems 2.1 and 2.3. As in [6], [1], [10], the convergence results will be obtained in two steps. Indeed, a weak convergence theorem can be obtained first by using the maximum principle property. Next, some further estimates and the analysis of the particular structure of the equations lead to the $C^1([0,1])$ convergence of the potentials and to the different properties of the limit models $\mathcal{F}_{\rm CL}^{\bf ni}$ and $\mathcal{F}_{\rm CL}^{\bf i}$. Finally in a third subsection we prove the space-charge limitation of current (2.3), (2.5).

4.1. A priori estimates and convergence result

Let $(f^\varepsilon, \varphi^\varepsilon, a^\varepsilon)$ be a solution of (1.2)-(1.10) given by Theorem 1.1. The function g being only defined for $p_x > 0$, one can write

$$g(p_x, p_y) = g_1(\mathbf{p}^2, p_y).$$

We assume that the function g is compactly supported. On the support of g, there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\lambda_1 \epsilon(\mathbf{p}) \leq \mathbf{p}^2 \leq \lambda_2 \epsilon(\mathbf{p}).$$

Setting

$$G(X,Y) = \sup_{\lambda_1 X \leq X' \leq \lambda_2 X} g_1(X', Y),$$

there exist $M > 0$ and $C_0 > 0$ such that

$$\left\{ \begin{array}{l} 0 \leq g^\varepsilon(p_x, p_y) \leq \dfrac{1}{\varepsilon^3} G\left(\dfrac{\epsilon(\mathbf{p})}{\varepsilon^2}, \dfrac{p_y}{\varepsilon}\right) \\ 0 \leq G(X,Y) \leq M \\ G(X,Y) = 0 \text{ if } X \geq C_0 \text{ or } |Y| \geq C_0. \end{array} \right. \tag{4.1}$$

For $x \in [0,1]$ we denote

$$\mathcal{V}^\varepsilon(x) = \left\{ \mathbf{p} \in \mathbb{R}^2 : \quad |\epsilon(\mathbf{p}) - \varphi^\varepsilon(x)| \leq C_0\, \varepsilon^2, \quad |p_y - a^\varepsilon(x)| \leq C_0\, \varepsilon \right\}.$$

Hence from (1.14) and (4.1) we deduce the

Lemma 4.1 *We have*

$$0 \leq f^\varepsilon \leq \frac{M}{\varepsilon^3} \tag{4.2}$$

and

$$\forall x \in [0,1], \quad \mathrm{Supp}\,(f^\varepsilon(x,.)) \subset \mathcal{V}^\varepsilon(x). \tag{4.3}$$

Next we set

$$\theta^\varepsilon(x) = (1+\varphi^\varepsilon(x))^2 - (a^\varepsilon(x))^2 - 1.$$

In this paper C denotes a generic constant independent of the small parameter ε.

Lemma 4.2 *The electric and magnetic potentials verify*

$$-C_0\,\varepsilon^2 \le \varphi^\varepsilon(x) \le \varphi_L, \tag{4.4}$$

$$-C\varepsilon \le a^\varepsilon(x) \le a_L. \tag{4.5}$$

Proof. The second inequality of (4.4) comes from the convexity of φ^ε. To prove the first one, remark that, by the definition of $\mathcal{V}^\varepsilon(x)$ and (4.3) we have

$$\varphi^\varepsilon(x_0) < -C_0\varepsilon^2 \Longrightarrow \mathcal{V}^\varepsilon(x_0) = \emptyset \Longrightarrow \frac{d^2\varphi^\varepsilon}{dx^2}(x_0) = n^\varepsilon(x_0) = 0. \tag{4.6}$$

Let $\varphi^\varepsilon(x_0) = \min\limits_{[0,1]}\varphi^\varepsilon$. If $x_0 = 0$ then (4.4) is proved. Otherwise, we have $0 < x_0 < 1$, thus $\varphi^{\varepsilon\prime}(x_0) = 0$. Therefore if $\varphi^\varepsilon(x_0) < -C_0\varepsilon^2$ then we deduce from (4.6) that $\varphi^\varepsilon \equiv \varphi^\varepsilon(x_0)$ on $[0,1]$, which is impossible.

By (1.4) and (4.3) we have

$$\frac{d^2a^\varepsilon}{dx^2}(x) = \int_{\mathcal{V}^\varepsilon(x)} \frac{p_y f^\varepsilon(x,\mathbf{p})}{\sqrt{1+\mathbf{p}^2}}d\mathbf{p} = a^\varepsilon\int_{\mathcal{V}^\varepsilon(x)}\frac{1}{\sqrt{1+\mathbf{p}^2}}f^\varepsilon d\mathbf{p} + \int_{\mathcal{V}^\varepsilon(x)}\frac{(p_y - a^\varepsilon)}{\sqrt{1+\mathbf{p}^2}}f^\varepsilon d\mathbf{p}.$$

Let

$$M_\varepsilon = \int_{\mathcal{V}^\varepsilon(x)}\frac{1}{\sqrt{1+\mathbf{p}^2}}f^\varepsilon d\mathbf{p} \ge 0.$$

From the definition of $\mathcal{V}^\varepsilon$ we deduce

$$\frac{d^2a^\varepsilon}{dx^2} = (a^\varepsilon + m_\varepsilon C_0\varepsilon)\,M_\varepsilon, \tag{4.7}$$

where $|m_\varepsilon| \le 1$. Hence a^ε is convex as soon as $a^\varepsilon > C_0\varepsilon$, which implies the second inequality of (4.5). (4.7) also enables to write

$$a^\varepsilon(x_0) < -C_0\varepsilon \Longrightarrow \frac{d^2a^\varepsilon}{dx^2}(x_0) \le 0.$$

Therefore, if $a^\varepsilon(x_0) = \min\limits_{[0,1]}a^\varepsilon$ and $a^\varepsilon(x_0) < -C_0\varepsilon$ then one can prove that $a^\varepsilon \equiv a^\varepsilon(x_0)$ on $[0,1]$, which is impossible. $\square$

Lemma 4.3 *For $x \in [0,1]$, $\mathcal{V}^\varepsilon(x)$ verifies*

$$\mathcal{V}^\varepsilon(x) \subset \Big\{\mathbf{p}\in\mathbb{R}^2 : \quad \begin{aligned} |p_x^2 - \theta^\varepsilon(x)| &\le K\varepsilon; \\ |p_y - a^\varepsilon(x)| &\le C_0\,\varepsilon \Big\}, \end{aligned} \tag{4.8}$$

$$\operatorname{meas}(\mathcal{V}^\varepsilon(x)) \le C\varepsilon^2, \tag{4.9}$$

$$\int_{\mathcal{V}^\varepsilon(x)} |p_x|\, d\mathbf{p} \le C\varepsilon^3, \tag{4.10}$$

$$\int_{\mathcal{V}^\varepsilon(x)} p_x^2\, d\mathbf{p} \le C\varepsilon^3 \sqrt{(\theta^\varepsilon(x) + K\varepsilon)^+} \tag{4.11}$$

where $(.)^+ = \max(.,0)$

Proof. (4.8) is immediate: it suffices to replace ϵ in the definition of $\mathcal{V}^\varepsilon$ by $\sqrt{\mathbf{p}^2+1}-1$ and to use the estimates (4.4) and (4.5).

Remark now that there exists $C > 0$ such that

$$(p_x, p_y) \in \mathcal{V}^\varepsilon(x) \Longrightarrow (|p_x|, p_y) \in \widetilde{\mathcal{V}}_+^\varepsilon(x),$$

where we note

$$\widetilde{\mathcal{V}}_+^\varepsilon(x) = \{\mathbf{p} \in \mathbb{R}_+^2 : \quad |p_y - a^\varepsilon(x)| \le C_0\varepsilon^2, \quad |\mathbf{p}^2 - ((1+\varphi^\varepsilon)^2 - 1)| \le C\varepsilon^2\}.$$

For (4.9) we change to polar coordinates:

$$\int_{\mathcal{V}^\varepsilon(x)} dp_x\, dp_y \le 2\int_{\widetilde{\mathcal{V}}_+^\varepsilon(x)} d\mathbf{p} \le 2\pi \int_{(1+\varphi^\varepsilon)^2-1-C\varepsilon^2}^{(1+\varphi^\varepsilon)^2-1+C\varepsilon^2} d(\mathbf{p}^2) = 4\pi C\varepsilon^2.$$

For (4.10) we apply the change of variable

$$(p_x, p_y) \to (X, Y) = (\mathbf{p}^2 - (1+\varphi^\varepsilon)^2 + 1, p_y - a^\varepsilon(x))$$

for $p_x > 0$ to obtain

$$\int_{\mathcal{V}^\varepsilon(x)} |p_x|\, d\mathbf{p} \le 2\int_{\widetilde{\mathcal{V}}_+^\varepsilon(x)} p_x\, d\mathbf{p} \le 2\int_{X=-C\varepsilon^2}^{C\varepsilon^2} \int_{Y=-C_0\varepsilon}^{C_0\varepsilon} dX\, dY = 8CC_0\varepsilon^3.$$

Finally (4.11) is deduced from (4.8) and (4.10). $\square$

This lemma directly implies the following corollary:

Corollary 4.4 *We have*

$$|n^\varepsilon(x)| \le \frac{C}{\varepsilon}; \quad |j_y^\varepsilon(x)| \le \frac{C}{\varepsilon}; \quad \int_{\mathbb{R}^2} |p_x| f^\varepsilon\, d\mathbf{p} \le C.$$

Moreover there exists $K > 0$ *such that*

$$\forall x \in [0,1], \quad \theta^\varepsilon(x) \le -K\varepsilon \Longrightarrow f^\varepsilon(x,.) \equiv 0. \tag{4.12}$$

Lemma 4.5 *For ε small enough, we have*

$$-C\sqrt{\varepsilon} \leq {\varphi^\varepsilon}'(0) \leq \varphi_L; \tag{4.13}$$

$$-C \leq {a^\varepsilon}'(0) \leq C\varepsilon + a_L. \tag{4.14}$$

Proof. The second inequality of (4.13) is a consequence of the convexity of φ^ε. For the first one, we use the estimate of n^ε given by Corollary 4.4 and Poisson's equation (1.3). After two integrations we obtain

$$\varphi^\varepsilon(x) \leq \frac{C}{2\varepsilon}x^2 + {\varphi^\varepsilon}'(0)\,x;$$

thus, by (4.4),

$$\forall x \in [0,1], \quad \frac{C}{2\varepsilon}x^2 + {\varphi^\varepsilon}'(0)\,x + C_0\varepsilon^2 \geq 0. \tag{4.15}$$

Assume ${\varphi^\varepsilon}'(0) < 0$ and that the discriminant of the left hand side of (4.15) is nonnegative. Hence this trinomial admits two positive roots $x_1 \leq x_2$ such that $x_1x_2 = 2\varepsilon^3C_0/C$. For ε small enough this implies $x_1 < 1$, which leads to a contradiction with (4.15). Therefore we have

$${\varphi^\varepsilon}'(0) \geq 0 \quad \text{or} \quad |{\varphi^\varepsilon}'(0)| < \left(\frac{CC_0\varepsilon}{2}\right)^{1/2},$$

and (4.13) is proved. Similarly, with the estimate of j_y^ε given by Corollary 4.4, Ampere's equation and (4.5) we can show the left inequality of (4.14). It remains to prove its right inequality.

Let $x_\varepsilon = \inf\{x : a^\varepsilon(x) = C_0\varepsilon\}$. By (4.7) and Corollary 4.4, on $[0, x_\varepsilon]$ we have

$$|{a^\varepsilon}''| \leq C\varepsilon M_\varepsilon \leq C\varepsilon n^\varepsilon \leq C_1.$$

Hence

$${a^\varepsilon}'(x) \geq {a^\varepsilon}'(0) - C_1 x \quad \text{on } [0, x_\varepsilon]; \tag{4.16}$$

after integration, since $a^\varepsilon(x) \leq C_0\varepsilon$ on $[0, x_\varepsilon]$, we obtain

$$\frac{C_1}{2}x^2 - {a^\varepsilon}'(0)x + C_0\varepsilon \geq 0 \quad \text{on } [0, x_\varepsilon]. \tag{4.17}$$

Without loss of generality we can assume that ${a^\varepsilon}'(0) \geq a_L$, thus for ε small enough this trinomial admits two positive roots. Moreover to fulfill (4.17) x_ε must be smaller than these two roots, which yields, after a calculation,

$$x_\varepsilon \leq \frac{2C_0\varepsilon}{{a^\varepsilon}'(0)}. \tag{4.18}$$

Therefore in (4.16) it gives

$${a^\varepsilon}'(x_\varepsilon) \geq {a^\varepsilon}'(0) - \frac{2C_1C_0\varepsilon}{{a^\varepsilon}'(0)}. \tag{4.19}$$

Thanks to the definition of x_ε, we have ${a^\varepsilon}'(x_\varepsilon) \geq 0$. Recall now that by (4.7) a^ε is convex if $a^\varepsilon \geq C_0\varepsilon$. Consequently we have $a^\varepsilon(x) \geq C_0\varepsilon$ on $[x_\varepsilon, 1]$, and a^ε is convex on this interval:

$$ {a^\varepsilon}'(x_\varepsilon) \leq \frac{a_L - C_0\varepsilon}{1 - x_\varepsilon}, $$

thus with (4.18) and (4.19) we get

$$ ({a^\varepsilon}'(0))^2 - (a_L + C_0\varepsilon){a^\varepsilon}'(0) - 2C_1C_0\varepsilon \leq -2C_1C_0x_\varepsilon\,\varepsilon < 0. $$

By writing that ${a^\varepsilon}'(0)$ is smaller than the largest root of this trinomial, we obtain finally the right inequality of (4.14). $\square$

Lemma 4.6 *The following estimates hold, for ε small enough:*

$$ |h^\varepsilon| \leq C, \tag{4.20} $$

$$ \left\| \frac{d\varphi^\varepsilon}{dx} \right\|_{L^\infty(0,1)} + \left\| \frac{da^\varepsilon}{dx} \right\|_{L^\infty(0,1)} \leq C, \tag{4.21} $$

$$ \|\varphi^\varepsilon\|_{W^{2,1}(0,1)} + \|a^\varepsilon\|_{W^{2,1}(0,1)} + \|f^\varepsilon\|_{L^1((0,1\times\mathbb{R}^2))} \leq C. \tag{4.22} $$

Proof. Let us first estimate h^ε. Setting

$$ k^\varepsilon = \int_{\mathbb{R}^2} v_x p_x f^\varepsilon(x, p_x, p_y) d\mathbf{p}, $$

we have

$$ 0 \leq k^\varepsilon = \int_{\mathbb{R}^2} \frac{p_x^2}{\sqrt{1+\mathbf{p}^2}} f^\varepsilon(x, p_x, p_y) d\mathbf{p} \leq \int_{\mathbb{R}^2} p_x^2 f^\varepsilon(x, p_x, p_y) d\mathbf{p}, $$

thus by Lemma 4.1 and (4.11) we have

$$ \|k^\varepsilon\|_{L^\infty(0,1)} \leq C. \tag{4.23} $$

Moreover, with this function k^ε, (1.16) is written

$$ h^\varepsilon = k^\varepsilon(x) \; - \frac{1}{2}\left[\left(\frac{d\varphi^\varepsilon}{dx}\right)^2 - \left(\frac{da^\varepsilon}{dx}\right)^2 \right]. \tag{4.24} $$

Hence Lemma 1.2, (4.13), (4.14) and (4.23) enable us to estimate $h^\varepsilon \equiv h^\varepsilon(0)$ and (4.20) is proved.

Next (4.20) and (4.23) imply

$$ \left[\left(\frac{d\varphi^\varepsilon}{dx}\right)^2 - \left(\frac{da^\varepsilon}{dx}\right)^2 \right] \text{is bounded in } L^\infty(0,1). \tag{4.25} $$

To prove that $\varphi^{\varepsilon\prime}$ and $a^{\varepsilon\prime}$ are separately bounded we derive another relation between these two quantities. On $\mathcal{V}^\varepsilon(x)$ we have

$$\frac{a^\varepsilon - C_0\varepsilon}{1+\varphi^\varepsilon + C_0\varepsilon^2} \le v_y \le \frac{a^\varepsilon + C_0\varepsilon}{1+\varphi^\varepsilon - C_0\varepsilon^2},$$

thus

$$\left| v_y - \frac{a^\varepsilon}{1+\varphi^\varepsilon}\right| \le C\varepsilon.$$

This implies, in view of Lemma 4.1,

$$\begin{aligned} \left|(1+\varphi^\varepsilon)(x)a^{\varepsilon\prime\prime}(x) - a^\varepsilon(x)\varphi^{\varepsilon\prime\prime}(x)\right| &= \left|\int_{\mathcal{V}^\varepsilon(x)} \left((1+\varphi^\varepsilon(x))v_y - a^\varepsilon(x)\right) f^\varepsilon(x,\mathbf{p})d\mathbf{p}\right| \\ &\le C\varepsilon(1+\varphi^\varepsilon(x))n^\varepsilon(x). \\ &\le C\varepsilon(1+\varphi^\varepsilon)\varphi^{\varepsilon\prime\prime} + C\varepsilon\left(\varphi^{\varepsilon\prime}\right)^2. \end{aligned} \tag{4.26}$$

An integration yields

$$\left|(1+\varphi^\varepsilon(x))a^{\varepsilon\prime}(x) - a^\varepsilon(x)\varphi^{\varepsilon\prime}(x) - a^{\varepsilon\prime}(0)\right| \le C\varepsilon\left((1+\varphi^\varepsilon(x))\varphi^{\varepsilon\prime}(x) - \varphi^{\varepsilon\prime}(0)\right). \tag{4.27}$$

Since φ^ε is convex we have $\varphi^{\varepsilon\prime}(1) \ge \varphi_L$. Writing (4.27) at $x=1$ and dividing it by $(1+\varphi_L)\varphi^{\varepsilon\prime}(1)$ gives, with (4.13),

$$\left|\frac{a^{\varepsilon\prime}(1)}{\varphi^{\varepsilon\prime}(1)} - \frac{a_L}{1+\varphi_L} - \frac{a^{\varepsilon\prime}(0)}{(1+\varphi_L)\varphi^{\varepsilon\prime}(1)}\right| \le C\varepsilon. \tag{4.28}$$

Assume that $\varphi^{\varepsilon\prime}(1) \to +\infty$. By (4.25) we also have $a^{\varepsilon\prime}(1) \to +\infty$ and

$$a^{\varepsilon\prime}(1)/\varphi^{\varepsilon\prime}(1) \to 1. \tag{4.29}$$

Therefore the left hand side of (4.28) tends to

$$1 - \frac{a_L}{1+\varphi_L}.$$

If $a_L \ne 1+\varphi_L$ this leads to a contradiction. Consider now the case $a_L = 1+\varphi_L$. We have $\theta_L = -1$ thus $\theta^\varepsilon < -K\varepsilon$ in a neighborhood of 1. Let $x_\varepsilon = \sup\{x : \theta^\varepsilon(x) = -K\varepsilon\}$. For $x \ge x_\varepsilon$ there holds $\theta^\varepsilon(x) \le -K\varepsilon$ thus by (4.12) $\varphi^{\varepsilon\prime\prime}(x) = a^{\varepsilon\prime\prime}(x) = 0$. Hence

$$\varphi^{\varepsilon\prime}(x_\varepsilon) = \varphi^{\varepsilon\prime}(1) \quad \text{and} \quad a^{\varepsilon\prime}(x_\varepsilon) = a^{\varepsilon\prime}(1).$$

Write (4.27) at this point x_ε and divide it by $(1+\varphi^\varepsilon(x_\varepsilon))\varphi^{\varepsilon\prime}(1)$:

$$\left|\frac{a^{\varepsilon\prime}(1)}{\varphi^{\varepsilon\prime}(1)} - \frac{a^\varepsilon(x_\varepsilon)}{1+\varphi^\varepsilon(x_\varepsilon)} - \frac{a^{\varepsilon\prime}(0)}{(1+\varphi^\varepsilon(x_\varepsilon))\varphi^{\varepsilon\prime}(1)}\right| \le C\varepsilon. \tag{4.30}$$

We still have (4.29) and

$$\frac{a^{\varepsilon\prime}(0)}{1+\varphi^\varepsilon(x_\varepsilon))\varphi^{\varepsilon\prime}(1)} \to 0.$$

Additionally $\theta^\varepsilon(x_\varepsilon) = -K\varepsilon$ implies

$$a^\varepsilon(x_\varepsilon) = \sqrt{(1+\varphi^\varepsilon(x_\varepsilon))^2 - 1 + K\varepsilon},$$

thus

$$\text{limsup}\frac{a^\varepsilon(x_\varepsilon)}{1+\varphi^\varepsilon(x_\varepsilon)} = \text{limsup}\sqrt{\left(1 - \frac{1-K\varepsilon}{(1+\varphi^\varepsilon(x_\varepsilon))^2}\right)} \leq \sqrt{1 - \frac{1}{(1+\varphi_L)^2}} < 1.$$

Therefore, as $\varepsilon \to 0$ the left hand side of (4.30) cannot tend to 0, which also leads to a contradiction. Hence, in both cases ${\varphi^\varepsilon}'(1)$ is bounded, thus by convexity

$$\|{\varphi^\varepsilon}'\|_{L^\infty(0,1)} \leq C$$

and from (4.25) we deduce (4.21).

To prove (4.22) it suffices to remark that

$$\|f^\varepsilon\|_{L^1((0,1)\times\mathbf{R}^2)} = \|n^\varepsilon\|_{L^1(0,1)} = \|{\varphi^\varepsilon}''\|_{L^1(0,1)} = |{\varphi^\varepsilon}'(1) - {\varphi^\varepsilon}'(0)| \leq C$$

and that

$$\|{a^\varepsilon}''\|_{L^1(0,1)} \leq C\|n^\varepsilon\|_{L^1(0,1)}.$$

□

This lemma implies the

Corollary 4.7 *After extraction of a subsequence and as $\varepsilon \to 0$ we have*

$$\begin{array}{l}
f^\varepsilon \to f \quad \textit{in } \mathcal{M}_b([0,1]\times\mathbb{R}^2) \textit{ weak } *\\
\varphi^\varepsilon \to \varphi \quad \textit{in } W^{1,\infty}(0,1) \textit{ weak } *\\
\qquad\qquad \textit{and in } W^{1,p}(0,1) \textit{ strong, for all } p<\infty\\
a^\varepsilon \to a \quad \textit{in } W^{1,\infty}(0,1) \textit{ weak } *\\
\qquad\qquad \textit{and in } W^{1,p}([0,1]) \textit{ strong, for all } 1<p<\infty\\
n^\varepsilon \to n = \varphi'' \quad \textit{in } \mathcal{M}_b([0,1]) \textit{ weak } *\\
j_y^\varepsilon \to j_y = a'' \quad \textit{in } \mathcal{M}_b([0,1]) \textit{ weak } *\\
k^\varepsilon \to k \quad \textit{in } L^\infty(0,1) \textit{ weak } *\\
j_x^\varepsilon \to j_x\\
h^\varepsilon \to h.
\end{array}$$

We can pass to the limit in (4.24) to obtain

$$h = k(x) - \frac{1}{2}\left[\left(\frac{d\varphi}{dx}(x)\right)^2 - \left(\frac{da}{dx}(x)\right)^2\right]. \tag{4.31}$$

Let $\beta^\varepsilon = {a^\varepsilon}'(0)$. After another extraction, by (4.14), we have $\beta^\varepsilon \to \beta$. In fact we deduce from (4.27) the

Lemma 4.8 *We have $\beta = a'(0)$. Moreover $\left[(1+\varphi^\varepsilon)a^{\varepsilon\prime} - a^\varepsilon\varphi^{\varepsilon\prime} - \beta^\varepsilon\right]$ converges uniformly on $[0,1]$ to*

$$(1+\varphi)a' - a\varphi' - \beta \equiv 0.$$

We also have obviously $\theta^\varepsilon \to \theta$ in every $W^{1,p}([0,1])$ strong, for $1 < p < \infty$. Remark finally that the C^0 convergence of the potentials and the estimates (4.4) and (4.5) enable us to show that the limit potentials verify (1.i), (2.i) and (1.ii), (2.ii). Moreover we can also pass to the limit in (4.26), in the sense of bounded measures to get the property (2.iv).

4.2. Stronger convergence and qualitative structure of the limit solution

We begin with the following lemma which gives a qualitative information on θ.

Lemma 4.9 *Assume that there exists $x_1 \in (0,1)$ such that $\theta(x_1) < 0$. Then $\theta < 0$ on $[x_1, 1]$.*

Proof. Let ε be small enough and $(x_0^\varepsilon, x_2^\varepsilon)$ be the largest interval around x_1 on which $\theta^\varepsilon < -K\varepsilon$ where K is the same constant as in (4.8). It is clear that $\theta^\varepsilon(x_0^\varepsilon) = -K\varepsilon$ and that $\theta^{\varepsilon\prime}(x_0^\varepsilon) \le 0$. The last inequality can be written

$$\varphi^{\varepsilon\prime}(x_0) \le a^{\varepsilon\prime}(x_0)\frac{a^\varepsilon(x_0)}{1+\varphi^\varepsilon(x_0)}. \tag{4.32}$$

and the equality $\theta^\varepsilon(x_0^\varepsilon) = -K\varepsilon$ implies

$$1 + \varphi^\varepsilon(x_0) = \sqrt{a^\varepsilon(x_0)^2 + 1 - K\varepsilon} > a^\varepsilon(x_0). \tag{4.33}$$

Moreover (4.12) implies that $\varphi^{\varepsilon\prime\prime} = a^{\varepsilon\prime\prime} = 0$ on $(x_0^\varepsilon, x_2^\varepsilon)$, thus $\theta^{\varepsilon\prime\prime}$ is constant and is equal to

$$\theta^{\varepsilon\prime\prime} = 2\left(\varphi^{\varepsilon\prime}\right)^2 - 2\left(a^{\varepsilon\prime}\right)^2.$$

We claim that (4.32) and (4.33) imply that $\theta^{\varepsilon\prime\prime} \le C\varepsilon$ on $(x_0^\varepsilon, x_2^\varepsilon)$ which implies with a continuation argument that θ is decreasing on $[x_1, 1]$ and is therefore negative on $[x_1, 1]$. In addition, our claim implies the following property

$$\exists C \in \mathbb{R}: \quad \forall\varepsilon > 0, \forall x_1 \in [0,1], \quad \theta^\varepsilon(x_1) \le -K\varepsilon \Longrightarrow \theta^\varepsilon(x) < C\varepsilon \text{ for } x \ge x_1. \tag{4.34}$$

Let us now prove the claim. We shall distinguish two cases.

Case 1 : $\varphi^{\varepsilon\prime}(x_0^\varepsilon) \le 0$. In this case, (4.13) implies $(\varphi^{\varepsilon\prime}(x_0^\varepsilon))^2 \le C\varepsilon$ leading to the desired estimate in view of the inequality $\theta^{\varepsilon\prime\prime} \le 2(\varphi^{\varepsilon\prime})^2$.

Case 2 : ${\varphi^\varepsilon}'(x_0^\varepsilon) \geq 0$. Here either $a^\varepsilon(x_0^\varepsilon) \geq 0$ and (4.32) and (4.33) lead to $({\varphi^\varepsilon}'(x_0^\varepsilon))^2 \leq ({a^\varepsilon}'(x_0^\varepsilon))^2$ which means ${\theta^\varepsilon}'' \leq 0$, or $a^\varepsilon(x_0^\varepsilon) \leq 0$ and (4.32) together with (4.5) lead to the estimate ${\varphi^\varepsilon}'(x_0^\varepsilon) \leq C\varepsilon$ which implies

$$
{\theta^\varepsilon}'' \leq C\varepsilon^2 \leq C\varepsilon.
$$

□

Lemma 4.9 thus directly implies the properties (1.*iii*) in the case $\theta_L > 0$ and (2.*iii*) in the case $\theta_L < 0$. We recall that the edge x^* of the sheath is defined by

$$
x^* = \sup\{x \in [0,1] : \quad \theta(x) \geq 0\}. \tag{4.35}
$$

Lemma 4.9 implies that $x^* < 1$ in the case $\theta_L < 0$ and $x^* = 1$ if $\theta_L > 0$. In addition, in the first case and for any $\alpha > 0$, since ${\varphi^\varepsilon}'$ and ${a^\varepsilon}'$ are constant on $[x^* + \alpha, 1]$ for ε small enough, φ^ε and a^ε converge respectively to φ and a in $C^1([x^* + \alpha, 1])$ strong. Therefore, condition (2.*vi*) is fulfilled.

Lemma 4.10 *Let $x_1 < x_2$ be such that $\theta > 0$ on (x_1, x_2). Then φ^ε and a^ε converge to φ and a in $W^{2,\infty}$ strong on every compact subset of (x_1, x_2). Moreover there exists a constant $\bar{\jmath}$ such that, on (x_1, x_2),*

$$
\varphi'' = \bar{\jmath}\, \frac{(1+\varphi)}{\sqrt{\theta}} \quad \text{and} \quad a'' = \bar{\jmath}\, \frac{a}{\sqrt{\theta}}, \tag{4.36}
$$

$$
h = \bar{\jmath}\sqrt{\theta(x)} - \frac{1}{2}\left[(\varphi')^2 - (a')^2\right]. \tag{4.37}
$$

Proof. Let $\alpha > 0$ be such that $x_1 + \alpha < x_2 - \alpha$. There exists $\eta > 0$ such that, on the interval $\mathrm{I} = [x_1 + \alpha, x_2 - \alpha]$ and for ε small enough, we have $\theta^\varepsilon > \eta$ and $\theta > \eta$. We recall that $\mathbb{R}^2_+ = \{\mathbf{p} \in \mathbb{R}^2 : \, p_x > 0\}$ and denote

$$
n_+^\varepsilon = \int_{\mathbb{R}^2_+} f^\varepsilon(x, \mathbf{p})\, d\mathbf{p}, \quad j_{x,+}^\varepsilon = \int_{\mathbb{R}^2_+} v_x f^\varepsilon(x, \mathbf{p})\, d\mathbf{p},
$$

$$
j_{y,+}^\varepsilon = \int_{\mathbb{R}^2_+} v_y f^\varepsilon(x, \mathbf{p})\, d\mathbf{p}, \quad k_+^\varepsilon = \int_{\mathbb{R}^2_+} v_x p_x f^\varepsilon(x, \mathbf{p})\, d\mathbf{p}.
$$

Integrate Vlasov's equation with respect to $\mathbf{p}$ on $(0, +\infty) \times \mathbb{R}$. Straightforward calculations lead to

$$
\frac{d}{dx} j_{x,+}^\varepsilon(x) = {\varphi^\varepsilon}'(x) \int_{-\infty}^{+\infty} f^\varepsilon(x, 0, p_y)\, dp_y - {a^\varepsilon}'(x) \int_{-\infty}^{+\infty} v_y\, f^\varepsilon(x, 0, p_y)\, dp_y. \tag{4.38}
$$

Of course this direct calculation is only formal, since the current $j_{x,+}^\varepsilon$ is not regular enough *a priori*. Nevertheless we can make it rigorous thanks to the weak formulation of the Vlasov equation, with suitable test functions. Such a proof is now standard and for it we refer to [6] Lemma 4.6, [1] Lemma 3.5 or [10] Lemma 3.8.

Next, by (4.8), if ε is small enough, we have $f^\varepsilon(x,\mathbf{p})=0$ for $x\in \mathrm{I}$ and $p_x<\sqrt{\eta}-C\varepsilon$. Hence

$$\frac{d}{dx}j^\varepsilon_{x,+}=0 \quad \text{on I.}$$

Thanks to Corollary 4.7 one can show that there exists a constant real number $j^I_{x,+}$ such that

$$j^\varepsilon_{x,+}\to j^I_{x,+} \quad \text{as } \varepsilon\to 0.$$

Since, for $x\in \mathrm{I}$, on $\mathcal{V}^\varepsilon(x)$ we have

$$\left|p_x-\sqrt{\theta^\varepsilon}\right|\le C\varepsilon, \quad \left|\sqrt{1+\mathbf{p}^2}-(1+\varphi^\varepsilon)\right|\le C\varepsilon^2, \quad |p_y-a^\varepsilon|\le C\varepsilon, \quad \theta^\varepsilon>\eta,$$

we deduce, for ε small enough,

$$\left|\frac{\sqrt{1+\mathbf{p}^2}}{p_x}-\frac{1+\varphi^\varepsilon}{\sqrt{\theta^\varepsilon}}\right|\le C\varepsilon \quad \text{and} \quad \left|\frac{p_y}{p_x}-\frac{a^\varepsilon}{\sqrt{\theta^\varepsilon}}\right|\le C\varepsilon,$$

This enables us to write

$$\left|k^\varepsilon_+(x)-j^\varepsilon_{x,+}\sqrt{\theta^\varepsilon}\right|=\left|\int_{\mathbb{R}^2_+}\left(p_x-\sqrt{\theta^\varepsilon}\right)v_x f^\varepsilon(x,\mathbf{p})\,d\mathbf{p}\right|\le C\varepsilon j^\varepsilon_{x,+},$$

$$\left|n^\varepsilon_+(x)-j^\varepsilon_{x,+}\frac{1+\varphi^\varepsilon}{\sqrt{\theta^\varepsilon}}\right|=\left|\int_{\mathbb{R}^2_+}\left(\frac{\sqrt{1+\mathbf{p}^2}}{p_x}-\frac{1+\varphi^\varepsilon}{\sqrt{\theta^\varepsilon}}\right)v_x f^\varepsilon(x,\mathbf{p})\,d\mathbf{p}\right|\le C\varepsilon j^\varepsilon_{x,+},$$

and

$$\left|j^\varepsilon_{y,+}(x)-j^\varepsilon_{x,+}\frac{a^\varepsilon}{\sqrt{\theta^\varepsilon}}\right|=\left|\int_{\mathbb{R}^2_+}\left(\frac{p_y}{p_x}-\frac{a^\varepsilon}{\sqrt{\theta^\varepsilon}}\right)v_x f^\varepsilon(x,\mathbf{p})\,d\mathbf{p}\right|\le C\varepsilon j^\varepsilon_{x,+}.$$

Therefore

$$k^\varepsilon_+(x)\to j^I_{x,+}\sqrt{\theta^\varepsilon}, \quad n^\varepsilon_+(x)\to j^I_{x,+}\frac{1+\varphi^\varepsilon}{\sqrt{\theta^\varepsilon}}, \quad j^\varepsilon_{y,+}(x)\to j^I_{x,+}\frac{a^\varepsilon}{\sqrt{\theta^\varepsilon}} \quad \text{uniformly on I.}$$

Obviously if we set $\mathbb{R}^2_-=\mathbb{R}^2\backslash\mathbb{R}^2_+$ and

$$n^\varepsilon_-=\int_{\mathbb{R}^2_-}f^\varepsilon(x,\mathbf{p})\,d\mathbf{p}=n^\varepsilon-n^\varepsilon_+, \quad j^\varepsilon_{x,-}=\int_{\mathbb{R}^2_-}v_x f^\varepsilon(x,\mathbf{p})\,d\mathbf{p}=j^\varepsilon_x-j^\varepsilon_{x,-},$$

$$j^\varepsilon_{y,-}=\int_{\mathbb{R}^2_-}v_y f^\varepsilon(x,\mathbf{p})\,d\mathbf{p}=j^\varepsilon_y-j^\varepsilon_{y,-}, \quad k^\varepsilon_-=\int_{\mathbb{R}^2_-}v_x p_x f^\varepsilon(x,\mathbf{p})\,d\mathbf{p}=k^\varepsilon-k^\varepsilon_+,$$

the same arguments allow to show that the function $j^\varepsilon_{x,-}$ is constant on I and converges to $j^I_{x,-}=j_x-j^I_{x,+}$. Moreover we have

$$k^\varepsilon_-(x)\to -j^I_{x,-}\sqrt{\theta}, \quad n^\varepsilon_-(x)\to -j^I_{x,-}\frac{1+\varphi}{\sqrt{\theta}}, \quad j^\varepsilon_{y,-}(x)\to -j^I_{x,-}\frac{a}{\sqrt{\theta}} \quad \text{uniformly.}$$

If we set $\bar{\jmath} = j^I_{x,+} - j^I_{x,-}$, we get

$$k^\varepsilon \to \bar{\jmath}\sqrt{\theta}, \quad n^\varepsilon \to \bar{\jmath}\frac{1+\varphi}{\sqrt{\theta}}, \quad j^\varepsilon_y \to \bar{\jmath}\frac{a}{\sqrt{\theta}} \quad \text{uniformly on I}$$

and the result follows from (1.3), (1.4) and (4.31). $\square$

From this lemma we deduce different properties for the limit models:

- in the magnetically insulated case, we have (2.v). Moreover, we have $j^\varepsilon_x(x) = 0$ on $[x^* + \alpha, 1]$, for $\alpha > 0$ and ε small enough. Hence by Lemma 1.2 we have $j^\varepsilon_x \equiv 0$ on $[0,1]$, thus $j^I_{x,+} = -j^I_{x,-} = \bar{\jmath}/2$.

- in the noninsulated case, from the boundary condition (1.8) in 1 we deduce $\bar{\jmath}_p = j_x$ and the property (1.vi). Moreover, if we assume (1.iv), then Lemma 4.10 and (1.iii) imply (1.v).

Let us now introduce a new function, similar to γ for $\varepsilon = 0$. Let

$$x^\varepsilon_1 = \sup\{x \in [0,1] : \theta^\varepsilon > -1 \text{ on } [0,x]\}.$$

For $\theta_L > 0$ we have $x^\varepsilon_1 = 1$ and for $\theta_L < 0$ we have $x^\varepsilon_1 > x^*$, for ε small enough. On $[0, x^\varepsilon_1)$ we can define the function γ^ε by

$$\gamma^\varepsilon(x) = \mathrm{Log}\sqrt{\frac{1+\varphi^\varepsilon(x)+a^\varepsilon(x)}{1+\varphi^\varepsilon(x)-a^\varepsilon(x)}} = \mathrm{Argtanh}\left(\frac{a^\varepsilon}{1+\varphi^\varepsilon}\right).$$

Hence the function γ^ε is as regular as φ^ε and a^ε and converges to γ in $C^0([0,x^*])$. Moreover

$$\begin{cases} 1+\varphi^\varepsilon(x) = \sqrt{1+\theta^\varepsilon(x)}\cosh(\gamma^\varepsilon(x)), \\ a^\varepsilon(x) = \sqrt{1+\theta^\varepsilon(x)}\sinh(\gamma^\varepsilon(x)). \end{cases} \tag{4.39}$$

Lemma 4.11 *The function γ^ε converges to γ in $C^1([0,x^*])$ strong (where x^* is given by (4.35)) and we have*

$$\left|\gamma^{\varepsilon\prime}(x) - \frac{\beta^\varepsilon}{1+\theta^\varepsilon(x)}\right| \le C\varepsilon \quad \textit{and} \quad \gamma' = \frac{\beta}{1+\theta}. \tag{4.40}$$

Proof. By (4.39) we have

$$(1+\varphi^\varepsilon(x))a^{\varepsilon\prime}(x) - a^\varepsilon(x)\varphi^{\varepsilon\prime}(x) = \gamma^{\varepsilon\prime}(x)(1+\theta^\varepsilon(x)),$$

thus (4.27) implies the result. $\square$

Lemma 4.12 *The electric and magnetic potentials φ^ε and a^ε converge respectively to φ and a in $C^1([0,1])$.*

Proof. First remark that by Lemma 4.8 the $C^1([0,1])$ convergence of φ^ε implies the $C^1([0,1])$ convergence of a^ε. Moreover, $\varphi^{\varepsilon\prime}$ is increasing: thanks to Dini's theorem it suffices to show that $\varphi^{\varepsilon\prime}$ converges pointwise to φ' and that φ' is continuous to conclude. Let $\Omega = \{x \in [0,1] : \theta(x) > 0\}$. We decompose this set into a countable union of disjoint open intervals

$$\Omega = \bigcup_{i\in\mathcal{I}} (x_i, y_i).$$

Consider a given interval (x_i, y_i). There holds $\theta(x_i) = 0$ and, if $y_i \neq 1$, $\theta(y_i) = 0$. By Lemma 4.10, the C^1 convergences of φ^ε and a^ε hold on every $[x_i + \alpha, y_i - \alpha]$, $\alpha > 0$: $\varphi^{\varepsilon\prime}$ converges pointwise to the continuous function φ' on (x_i, y_i). In the case $\theta_L > 0$ this convergence also holds on a $[1-\alpha, 1]$. In the case $\theta_L < 0$ we have seen that this convergence holds on $(x^*, 1]$. Finally

$$\theta(x) \neq 0 \quad \Longrightarrow \quad \lim_{y\to x} \theta'(y) = \lim_{\varepsilon\to 0} \theta^{\varepsilon\prime}(x) = \theta'(x). \tag{4.41}$$

Next by (4.39) and (4.40), straightforward but lengthy calculations lead to

$$\left|\left[\left(\frac{d\varphi^\varepsilon}{dx}(x)\right)^2 - \left(\frac{da^\varepsilon}{dx}(x)\right)^2\right] - \left[\frac{\left(\theta^{\varepsilon\prime}(x)\right)^2 - 4(\beta^\varepsilon)^2}{4(1+\theta^\varepsilon(x))}\right]\right| \le C\varepsilon,$$

thus, by (4.24),

$$\forall x \in [0,1] \quad \left|\left(\theta^{\varepsilon\prime}(x)\right)^2 - 4(\beta^\varepsilon)^2 - 8(1+\theta^\varepsilon(x))(k^\varepsilon(x) - h^\varepsilon)\right| \le C\varepsilon. \tag{4.42}$$

Consider now the set $\Omega_c = [0,x^*]\backslash\Omega$ and assume that $\Omega_c\backslash\{0,x^*\}$ is not empty. We shall prove that $h = \beta^2/2$, then that $\theta^{\varepsilon\prime}$ converges to 0 in $C^1(\Omega_c)$ strong. $\Omega_c \cap (0,x^*) \neq \emptyset$ implies that there exists $z_0 \in (0,x^*)$ such that $\theta(z_0) = 0$. We shall now distinguish several cases.

- In the first one, there exist two points z_1 and z_2 such that

$$z_1 < z_0 < z_2, \quad \theta(z_1) > 0 \quad \text{and} \quad \theta(z_2) > 0.$$

Let z_ε be such that $\theta^\varepsilon(z_\varepsilon) = \inf_{[z_1,z_2]} \theta^\varepsilon$. We have $\theta^{\varepsilon\prime}(z_\varepsilon) = 0$ for ε small enough. Writing (4.42) at this point gives:

$$\left|-4(\beta^\varepsilon)^2 - 8(1+\theta^\varepsilon(z_\varepsilon))(k^\varepsilon(z_\varepsilon) - h^\varepsilon)\right| \le C\varepsilon. \tag{4.43}$$

$\theta^\varepsilon(z_\varepsilon) \le \theta^\varepsilon(z_0)$ implies $\theta^\varepsilon(z_\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover Lemma 4.1 and (4.11) enable us to obtain

$$\forall x \in [0,1] \quad 0 \le k^\varepsilon(x) \le \sqrt{(\theta^\varepsilon(x) + C\varepsilon)^+}. \tag{4.44}$$

Therefore $k^\varepsilon(z_\varepsilon) \to 0$ and we can pass to the limit in (4.43): $h = \beta^2/2$.

- Otherwise we have $\theta \equiv 0$ either on $[0, z_0]$ or on $[z_0, x^*]$. For instance we consider $\theta \equiv 0$ on $[0, z_0]$; the other case can be treated similarly. For every $\varepsilon > 0$, there exists a point z_ε such that

$$z_\varepsilon \in [0, z_0] \quad \text{and} \quad \theta^{\varepsilon\prime}(z_\varepsilon) = \frac{\theta^\varepsilon(z_0)}{z_0}.$$

We have $\theta^\varepsilon(z_\varepsilon) \to 0$ and $\theta^{\varepsilon\prime}(z_\varepsilon) \to \theta(z_0)/z_0 = 0$. Hence as above it suffices to write (4.42) at the point z_ε then to pass to the limit to get $h = \beta^2/2$.
Next (4.42) and (4.44) lead to

$$\forall x \in [0,1] \quad \left|\theta^{\varepsilon\prime}(x)\right|^2 \leq \left|8(1+\theta^\varepsilon)h^\varepsilon - 4(\beta^\varepsilon)^2\right| + 8|1+\theta^\varepsilon| \sqrt{(\theta^\varepsilon(x) + C\varepsilon)^+} + C\varepsilon. \quad (4.45)$$

Therefore, since $\theta^\varepsilon \to 0$ uniformly on Ω_c and $2h^\varepsilon - (\beta^\varepsilon)^2 \to 0$, we deduce that $\theta^{\varepsilon\prime} \to 0$ uniformly on Ω_c and, by Lemma 4.11 and (4.39), that

$$\begin{cases} \varphi^{\varepsilon\prime} \to \beta \sinh(\gamma) \\ a^{\varepsilon\prime} \to \beta \cosh(\gamma) \end{cases} \quad \text{uniformly on } \Omega_c.$$

Hence to conclude the proof under the assumption $\Omega_c \cap (0, x^*) \neq \emptyset$ it suffices to show that the pointwise limit φ' is continuous. Passing to the limit in (4.45) yields

$$\forall x \in [0,1] \quad \left|\theta'(x)\right|^2 \leq 4\,\beta^2\,|\theta| \, + 8\,|1+\theta|\,\sqrt{\theta^+}.$$

Consequently, if $\theta(x) = 0$ then we have $\lim\limits_{x' \to x} \theta'(x') = 0$. θ' is continuous. With (4.41) it implies that φ' is continuous on $[0,1]$.

It remains now to consider the case where θ does not vanish on $(0, x^*)$. To conclude the proof, it suffices to show that

$$\lim_{x\to 0} \theta'(x) = \lim_{\varepsilon \to 0} \theta^{\varepsilon\prime}(0) \quad \text{and, if } x^* < 1, \quad \lim_{x \to x^*} \theta'(x) = \lim_{\varepsilon\to 0} \theta^{\varepsilon\prime}(x^*). \quad (4.46)$$

To this aim, we notice that (4.42) and (4.44) already imply

$$\lim_{x\to 0} |\theta'(x)| = \lim_{\varepsilon \to 0} \left|\theta^{\varepsilon\prime}(0)\right| = 2\sqrt{\beta^2 - 2h}$$

and, if $x^* < 1$,

$$\lim_{x\to x^*} |\theta'(x)| = \lim_{\varepsilon \to 0} \left|\theta^{\varepsilon\prime}(x^*)\right| = 2\sqrt{\beta^2 - 2h}.$$

If $2h = \beta^2$ then we can conclude directly. In the case $2h < \beta^2$, we deduce from $\theta > 0$ on $(0, x^*)$ and $\theta < 0$ on $(x^*, 1)$, that we necessarily have

$$\lim_{x\to 0} \theta'(x) = -\lim_{x \to x^*} \theta'(x) = 2\sqrt{\beta^2 - 2h}.$$

Besides, $\theta^{\varepsilon\prime}(0) = 2\varphi^{\varepsilon\prime}(0) \geq -C\sqrt{\varepsilon}$. Therefore, the limit as ε goes to zero of $\theta^{\varepsilon\prime}(0)$ is nonnegative and equal to $2\sqrt{\beta^2 - 2h}$. The only thing left to show is that $\lim \theta^{\varepsilon\prime}(x^*) \leq 0$. We notice that if this was not true, there would exist a sequence $x^\varepsilon \in (x^*, 1)$ such that

$\theta^{\varepsilon\prime}(x^\varepsilon) = 0$ and $\lim \theta^\varepsilon(x^\varepsilon) = 0$ (take for x^ε the point where θ^ε achieves its maximum on $[x^*, 1]$). Evaluating (4.42) at $x = x^\varepsilon$ and passing to the limit $\varepsilon \to 0$ leads to $2h = \beta^2$.

$\square$

The first part of Theorem 2.3, concerning the convergence of $(\mathcal{S}^\varepsilon)$ in the case $\theta_L < 0$, is proved. In the remainder of this subsection we consider $\theta_L > 0$: we prove the property $(1.iv)$ and the L^1 convergences of n^ε and j_y^ε, which imply the $W^{2,1}$ convergences of φ^ε and a^ε. This will end the proof of the first part of Theorem 2.1, which states the convergence of $(\mathcal{S}^\varepsilon)$ to an element of $\mathcal{F}^{\mathbf{ni}}$.

Lemma 4.13 *Let $\theta_L > 0$. Then for $x \in [0,1]$ we have*

$$\theta^\varepsilon(x) > -K\varepsilon, \tag{4.47}$$

$$\theta^{\varepsilon\prime\prime}(x) \geq 2\left(\varphi^{\varepsilon\prime}(x)\right)^2 - 2\left(a^{\varepsilon\prime}(x)\right)^2 + \frac{j_x^\varepsilon}{\sqrt{\theta^\varepsilon(x) + K\varepsilon}}. \tag{4.48}$$

Proof. Since $\theta_L > 0$, (4.34) directly implies (4.47). Moreover, since we have

$$\theta^{\varepsilon\prime\prime}(x) = 2\left(\varphi^{\varepsilon\prime}(x)\right)^2 - 2\left(a^{\varepsilon\prime}(x)\right)^2 + 2(1+\varphi^\varepsilon(x))n^\varepsilon(x) - 2a^\varepsilon(x)j_y^\varepsilon(x),$$

the proof of (4.48) reduces to estimate

$$\begin{aligned}\mathcal{J}^\varepsilon(x) &= 2(1+\varphi^\varepsilon(x))n^\varepsilon(x) - 2a^\varepsilon(x)j_y^\varepsilon(x) \\ &= 2\int_{\mathbb{R}^2}\left[(1+\varphi^\varepsilon(x))\sqrt{1+\mathbf{p}^2} - v_y\sqrt{1+\mathbf{p}^2}\,a^\varepsilon(x)\right]\frac{1}{\sqrt{1+\mathbf{p}^2}}f^\varepsilon(x,\mathbf{p})\,d\mathbf{p}.\end{aligned}$$

On $\mathcal{V}^\varepsilon(x)$ we have

$$(1+\varphi^\varepsilon(x))\sqrt{1+\mathbf{p}^2} - v_y\sqrt{1+\mathbf{p}^2}\,a^\varepsilon(x) \geq (1+\varphi^\varepsilon(x))^2 - (a^\varepsilon(x))^2 - C\varepsilon = \theta^\varepsilon(x) + 1 - C\varepsilon,$$

Thus, for ε small enough and $x \in [0,1]$,

$$(1+\varphi^\varepsilon(x))\sqrt{1+\mathbf{p}^2} - v_y\sqrt{1+\mathbf{p}^2}\,a^\varepsilon(x) \geq 1/2.$$

Moreover on $\mathcal{V}^\varepsilon(x)$ we have $|p_x| \leq \sqrt{\theta^\varepsilon(x) + K\varepsilon}$. Therefore, if $\mathcal{V}_+^\varepsilon(x) = \{\mathbf{p} \in \mathcal{V}^\varepsilon(x) : p_x > 0\}$,

$$\begin{aligned}\mathcal{J}^\varepsilon(x) &\geq \int_{\mathcal{V}^\varepsilon(x)} \frac{1}{\sqrt{1+\mathbf{p}^2}} f^\varepsilon(x,\mathbf{p})\,d\mathbf{p} \\ &\geq \int_{\mathcal{V}_+^\varepsilon(x)} \frac{1}{\sqrt{1+\mathbf{p}^2}} f^\varepsilon(x,\mathbf{p})\,d\mathbf{p} \\ &\geq \int_{p_x=0}^{\sqrt{\theta^\varepsilon(x)+K\varepsilon}} \int_{p_y=-\infty}^{+\infty} \frac{p_x}{\sqrt{\theta^\varepsilon(x)+K\varepsilon}} \frac{1}{\sqrt{1+\mathbf{p}^2}} f^\varepsilon(x,\mathbf{p})\,d\mathbf{p} \\ &\geq \frac{1}{\sqrt{\theta^\varepsilon(x)+K\varepsilon}} j_{x,+}^\varepsilon \geq \frac{1}{\sqrt{\theta^\varepsilon(x)+K\varepsilon}} j_x^\varepsilon\end{aligned}$$

which shows (4.48). $\square$

Lemma 4.14 *Let $\theta_L > 0$ and $j_x^g > 0$. Then we have $j_x > 0$ and the set $\{x \in [0,1] : \theta(x) = 0\}$ is finite.*

Proof. The main tool of this proof is again the relation (4.42). Passing to the limit in this relation yields

$$\forall x \in [0,1] \quad (\theta'(x))^2 - 4\beta^2 - 8(1+\theta(x))(k(x)-h) = 0 \tag{4.49}$$

We show that $j_x > 0$ by a contradiction argument. Remark that, by Lemma 1.2 and the boundary condition at 1 (1.8), we have for all $\varepsilon > 0$

$$j_x^\varepsilon = j_{x,+}^\varepsilon(1) \geq 0, \quad j_{x,-}^\varepsilon(1) = 0. \tag{4.50}$$

Therefore, $j_x \geq 0$.

Assume $j_x = 0$ and let $x_m = \max\{x \in [0,1] : \theta(x) = 0\}$. By Lemma 4.10 we have $k(x) = \bar{\jmath}\sqrt{\theta(x)}$ on $(x_m, 1]$, with $\bar{\jmath} = j_x - 2j_{x,-} = -2j_{x,-}$. By (4.50) $j_{x,-} = 0$. Hence $\bar{\jmath} = 0$ and $k \equiv 0$ on $(x_m, 1]$, which yields in view of (4.49)

$$h = \frac{\beta^2}{2(1+\theta(x))} - \frac{(\theta'(x))^2}{8(1+\theta(x))} < \frac{\beta^2}{2}. \tag{4.51}$$

Besides, relation (4.38) can be rewritten as follows (again, this equation has to be understood in a weak sense, multiplied by a test function and integrated on an interval $[x_1, x_2]$):

$$\frac{d}{dx} j_{x,+}^\varepsilon(x) = \frac{1}{2(1+\varphi^\varepsilon(x))} \int_{-\infty}^{+\infty} \left(\theta^{\varepsilon\prime}(x) + \sigma_\varepsilon\right) f^\varepsilon(x, 0, p_y)\, dp_y, \tag{4.52}$$

where

$$\sigma_\varepsilon = 2a^{\varepsilon\prime}(x)\left(a^\varepsilon(x) - p_y \frac{1+\varphi^\varepsilon(x)}{\sqrt{1+p_y^2}}\right).$$

Moreover we deduce from (4.3), (4.8) and (4.21), that for $(x, p_y) \in \mathrm{Supp}(f^\varepsilon(.,0,.))$, we have

$$|\sigma_\varepsilon| < C\varepsilon \quad \text{and} \quad |\theta^\varepsilon(x)| \leq K\varepsilon.$$

Hence, for ε small enough,

$$\begin{cases} \text{if } |\theta^\varepsilon(x)| > K\varepsilon \text{ then } \dfrac{d}{dx} j_{x,+}^\varepsilon(x) = 0 \\ \text{if } \theta^{\varepsilon\prime}(x) \geq C\varepsilon \text{ then} \dfrac{d}{dx} j_{x,+}^\varepsilon(x) \geq 0. \end{cases} \tag{4.53}$$

We have seen above that $j_{x,+}^\varepsilon(1) = j_x^\varepsilon \to 0$ as $\varepsilon \to 0$. Thanks to the boundary condition in 0 (1.7), we have $j_{x,+}^\varepsilon(0) = j_x^g > 0$. For ε small enough, this implies

$$j_{x,+}^\varepsilon(0) > j_{x,+}^\varepsilon(1).$$

Hence $j^\varepsilon_{x,+}$ cannot be nondecreasing on the whole interval $[0,1]$, thus, by (4.53) and a contradiction argument, we have

$$\exists x_\varepsilon \in [0,1]: \quad \theta^{\varepsilon\prime}(x_\varepsilon) < C\varepsilon \quad \text{and} \quad \theta^\varepsilon(x_\varepsilon) \le K\varepsilon.$$

In fact we can choose x_ε such that $|\theta^{\varepsilon\prime}(x_\varepsilon)| \le C\varepsilon$. Indeed, if $\theta^{\varepsilon\prime}(x_\varepsilon) < -C\varepsilon$, by (4.47) there exists $x'_\varepsilon > x_\varepsilon$ such that $\theta^{\varepsilon\prime}(x'_\varepsilon) = 0$ and $|\theta^\varepsilon(x'_\varepsilon)| \le K\varepsilon$ (take for x'_ε the point at which θ^ε achieves its minimum on $[x_\varepsilon, 1]$).
Thanks to (4.44) we also have $|k^\varepsilon(x_\varepsilon)| \le C\varepsilon$. Writing (4.42) at $x = x_\varepsilon$ and passing to the limit $\varepsilon \to 0$ gives $h = \beta^2/2$, which is in contradiction with (4.51). This proves that $j_x > 0$.

To prove that $\{x : \theta(x) = 0\}$ is finite we use the same argument as in [1] or [10], using Lemma 4.13 and $j_x > 0$. Let $x_1 < x_2$ be two points where θ vanishes. By Lemma 4.13 and (4.21), for ε small enough, there exists $n_D > 0$ such that

$$\text{on } [x_1, x_2] \quad \theta^{\varepsilon\prime\prime}(x) \ge \frac{j_x}{2\sqrt{\theta^\varepsilon + K\varepsilon}} - n_D. \tag{4.54}$$

Hence, from Lemma 3.1 of [1], (4.54) and the fact that $\theta(x_1) = \theta(x_2) = 0$ simultaneously imply that

$$x_2 - x_1 \ge \frac{3\sqrt{6}\, j_x}{2\, n_D^{3/2}}.$$

This gives a positive lower bound for $x_2 - x_1$, thanks to $j_x > 0$; this is enough to conclude. $\square$

Lemma 4.15 *Let $\theta_L > 0$. Then n^ε and j^ε_y converge respectively to n and j_y in $L^1(0,1)$.*

Proof. We refer to [1] or [10] for the details of this proof. By Lemma 4.14 the singular points, i.e. the points where θ vanishes, are isolated, denoted $\{x_0, \dots, x_p\}$. To show that n^ε and j^ε_y are Cauchy sequences in $L^1(0,1)$ we consider separately the intervals $[x_k + \alpha, x_{k+1} - \alpha]$, $\alpha > 0$, where the densities converge uniformly and the intervals $[x_k - \alpha, x_k + \alpha,]$, where we use the Poisson's and Ampère's equations and Lemma 4.12 as in

$$\begin{aligned} \left| \int_{x_k-\alpha}^{x_k+\alpha} (n^{\varepsilon_1} - n^{\varepsilon_2}) \right| \le\; & \left| \frac{d\varphi^{\varepsilon_1}}{dx}(x_k + \alpha) - \frac{d\varphi^{\varepsilon_1}}{dx}(x_k - \alpha) \right| \\ & + \left| \frac{d\varphi^{\varepsilon_2}}{dx}(x_k + \alpha) - \frac{d\varphi^{\varepsilon_2}}{dx}(x_k - \alpha) \right|. \end{aligned}$$

Remark that in the case of magnetic insulation $\theta_L < 0$, and due to the possible infinite number of singular points, one cannot directly perform the same kind of proof. That is the reason why we only stated the C^1 convergences of the potentials in Theorem 2.3, which is nevertheless sufficient for the study of the limit model. $\square$

4.3. The Child-Langmuir regime

We prove the existence of a bound j_x^{max} such that

$$j_x^g \geq j_x^{max} \quad \Longrightarrow \quad \varphi'(0) = 0. \tag{4.55}$$

This ends the proofs of Theorems 2.1 and 2.3.

Lemma 4.16 *Assume $\varphi'(0) > 0$. Then*

$$\begin{aligned} &\text{if } \theta_L > 0 \text{ we have } \theta > 0 \text{ on } (0,1] \text{ and } j_x = \bar{\jmath} = j_x^g \\ &\text{if } \theta_L < 0 \text{ we have } \theta > 0 \text{ on } (0,x^*) \text{ and } \bar{\jmath} \geq 2j_x^g. \end{aligned}$$

Proof. By (4.49), if $\theta'(0) > 0$ then we have $h < \beta^2/2$. Besides, in the proof of Lemma 4.12 we showed that if $\Omega_c \backslash \{0, x^*\} \neq \emptyset$ then $h = \beta^2/2$. This implies that θ does not vanish on $(0, x^*)$. It remains now to determine the value of $\bar{\jmath}$. To this aim, we first deduce from the C^1 regularity of θ the existence of $x_0 > 0$ and $\alpha > 0$ such that

$$\theta' > \alpha \quad \text{on } [0, x_0]. \tag{4.56}$$

If $\theta_L > 0$, there exists $C > 0$ such that

$$\theta > C \quad \text{on } [x_0, 1].$$

Since $\theta^\varepsilon \to \theta$ in $C^1([0,1])$, for ε small enough, we have

$$\theta^{\varepsilon\prime} > \alpha/2 \text{ on } [0, x_0] \text{ and } \theta^\varepsilon > K\varepsilon \quad \text{on } [x_0, 1].$$

Hence from (4.53) we deduce

$$\frac{d}{dx} j_{x,+}^\varepsilon(x) \geq 0 \text{ on } [0,1]. \tag{4.57}$$

We complete the proof in this case thanks to the boundary conditions of the Vlasov equation in 0 and 1. Indeed, (1.7) implies $j_x^\varepsilon \leq j_{x,+}^\varepsilon(0) = j_x^g$ and (1.8) implies $j_{x,+}^\varepsilon(1) = j_x^\varepsilon$. Therefore from (4.57) we deduce $j_x^\varepsilon = j_x^g$.

If $\theta_L < 0$, we use (4.53) and (4.56) again to deduce that, for ε small enough, $j_{x,+}^\varepsilon$ is non decreasing on $[0, x_0]$, thus

$$j_{x,+}^\varepsilon \geq j_x^g \text{ on } [0, x_0].$$

Consequently we have $\bar{\jmath} = 2 \lim_{\varepsilon \to 0} j_{x,+}^\varepsilon \geq 2 j_x^g$. □

Lemma 4.17 *Assume $\theta'(0) > 0$. Then, if $\theta_L \neq 1$, there exists j_x^{max}, only depending on the data φ_L and a_L, such that $j_x^g \leq j_x^{max}$.*

Proof. *Case of a weak magnetic field, $\theta_L > 0$.*
Since $\theta'(0) > 0$, by Lemma 4.16 and the property (1.v), for $x > 0$ we have

$$\varphi'' = j_x^g \frac{1+\varphi}{\sqrt{\theta}} \geq j_x^g.$$

Hence

$$\forall x \in [0,1] \quad j_x^g \frac{x^2}{2} \leq \varphi(x) - x\varphi'(0) \leq \varphi(x),$$

which gives, at $x = 1$,

$$j_x^g \leq 2\varphi_L.$$

Case of a strong magnetic field, $\theta_L < 0$.
If $\beta = 0$ then by (4.40) $\gamma \equiv 0$ on $[0, x^*]$ thus $a \equiv 0$, which is impossible since $a(x^*)^2 = (1+\varphi(x^*))^2 - 1$ and $\varphi > 0$ on $(0,1]$. Therefore $\beta > 0$ and since $\theta'(x^*) \leq 0$, it can be written $\theta'(x^*) = -2\alpha\beta$ with $\alpha \geq 0$.

In analogy with the proof of Lemma 3.5, where α was equal to zero, we obtain the following identities

$$\begin{cases} 1 + \varphi(x^*) = \cosh(\gamma^*), \quad a(x^*) = \sinh(\gamma^*), \\ \varphi'(x^*) = \beta\left[\sinh(\gamma^*) - \alpha\cosh(\gamma^*)\right], \\ a'(x^*) = \beta\left[\cosh(\gamma^*) - \alpha\sinh(\gamma^*)\right]. \end{cases} \tag{4.58}$$

where $\gamma^* = \gamma(x^*)$. Moreover, φ and a being affine functions on $[x^*, 1]$, we have

$$\frac{\varphi'(x^*)}{a'(x^*)} = \frac{1 + \varphi_L - \cosh(\gamma^*)}{a_L - \sinh(\gamma^*)}. \tag{4.59}$$

From (4.58) and (4.59) one can prove that $X = \exp(\gamma^*)$ solves the following equation, analogous to (3.44):

$$P_\alpha(X) := (1-\alpha)(a_L - \varphi_L - 1)X^2 + 2X - (1+\alpha)(1 + a_L + \varphi_L) = 0. \tag{4.60}$$

Recall that

$$X_0 = \frac{1 + a_L + \varphi_L}{1 + \sqrt{-\theta_L}}$$

is the smallest positive solution of $P_0(X) = 0$. We will now prove the claim

$$\forall X_\alpha \geq 1, \qquad P_\alpha(X_\alpha) = 0 \quad \Longrightarrow \quad X_\alpha \geq X_0. \tag{4.61}$$

The reduced discriminant of (4.60) is

$$\Delta = -\theta_L + \alpha^2(\theta_L + 1).$$

Assume $a_L \geq (1 + \varphi_L)$. This is equivalent to $\theta_L + 1 \leq 0$. If $\Delta \geq 0$ the solutions of $P_\alpha(X) = 0$ are

$$\frac{(1+\alpha)(1 + a_L + \varphi_L)}{1 \pm \sqrt{-\theta_L + \alpha^2(\theta_L + 1)}}.$$

Hence the smallest positive solution is

$$\frac{(1+\alpha)(1+a_L+\varphi_L)}{1+\sqrt{-\theta_L+\alpha^2(\theta_L+1)}} \geq \frac{(1+a_L+\varphi_L)}{1+\sqrt{-\theta_L}} = X_0,$$

and (4.61) is proved in this case.
Assume now $a_L < (1+\varphi_L)$. In the proof of Lemma 3.5, we have seen that the equation $P_0(X) = 0$ admits two solutions larger than 1: X_0 and

$$X_0' = \frac{1+a_L+\varphi_L}{1-\sqrt{-\theta_L}}.$$

Straightforward calculations give

$$P_\alpha(1) = -2\varphi_L - 2\alpha a_L < 0$$

$$P_\alpha(X_0) = 2\alpha\,[\,X_0 - (1+a_L+\varphi_L)] < 0$$

and

$$P_\alpha(X_0') = 2\alpha\,[\,X_0' - (1+a_L+\varphi_L)] > 0.$$

Therefore the equation $P_\alpha(X) = 0$ does not admit any solution between 1 and X_0: the claim (4.61) is proved.

From (4.61) we deduce $\gamma^* \geq \mathrm{Log} X_0$, thus

$$\gamma^* \geq \mathrm{Log}\left(\frac{1+a_L+\varphi_L}{1+\sqrt{-\theta}}\right),$$

and, by (4.40),

$$\beta x^* \geq \int_0^{x^*} \gamma'(x)\,dx \geq \mathrm{Log}\left(\frac{1+a_L+\varphi_L}{1+\sqrt{-\theta}}\right). \tag{4.62}$$

Next, by Lemma 4.16 and (2.v) we have $\theta > 0$ on $(0, x^*)$ and

$$\varphi'' = \bar{\jmath}\frac{1+\varphi}{\sqrt{\theta}}, \quad a'' = \bar{\jmath}\frac{a}{\sqrt{\theta}} \quad \text{on } (0, x^*). \tag{4.63}$$

Hence a is convex on $[0,1]$, thus $\beta = a'(0) \leq a_L$. This gives in view of (4.62)

$$x^* \geq \frac{1}{a_L}\mathrm{Log}\left(\frac{1+a_L+\varphi_L}{1+\sqrt{-\theta}}\right). \tag{4.64}$$

To conclude the proof, we proceed like in the weak magnetic field case and integrate twice the inequality $\varphi'' \geq \bar{\jmath}$ on $(0, x^*)$. This gives

$$\bar{\jmath} \leq \frac{2}{(x^*)^2}\varphi(x^*) \leq \frac{2}{(x^*)^2}\varphi_L. \tag{4.65}$$

To conclude, we recall that Lemma 4.16 implies $j_x^g \leq \bar{\jmath}/2$. Therefore, by (4.64) and (4.65), we have

$$j_x^g \leq \frac{\varphi_L (a_L)^2}{\left[\mathrm{Log}\left(\dfrac{1 + a_L + \varphi_L}{1 + \sqrt{-\theta}}\right)\right]^2}.$$

□

Finally, from Lemma 4.17, we deduce (4.55), which ends the proofs of Theorems 2.1 and 2.3.

References

[1] N. Ben Abdallah, *The Child-Langmuir regime for electron transport in a plasma including a background of positive ions*, Math. Models Methods Appl. Sci., **4**, 1994, pp. 409-438.

[2] N. Ben Abdallah, *Convergence of the Child-Langmuir asymptotics of the Boltzmann equation of semiconductors*, SIAM J. Math. Anal., vol. 27, **1**, 1996, pp. 92-109. (1996).

[3] N. Ben Abdallah and P. Degond, *The Child-langmuir Law for the Boltzmann equation of semiconductors*, SIAM J. Math. Anal., vol. 26, **2**, 1995, pp.364-398.

[4] N. Ben Abdallah, P. Degond, F. Méhats, *Mathematical models for magnetic insulation*, manuscript.

[5] J.M. Creedon, *Magnetic cutoff in high-current diodes*, J. Appl. Phys., **48**, No 3, 1977, pp. 1070-1077.

[6] P. Degond, S. Jaffard, F. Poupaud, P.-A. Raviart, *The Child-Langmuir Asymptotics of the Vlasov-Poisson Equation for Cylindrically or Spherically Symmetric Diodes. Parts 1 and 2*, Math. Methods Appl. Sci., **19**, 1996, pp. 287-312 and pp. 313-340.

[7] P. Degond, P.-A. Raviart, *An asymptotic analysis for the one-dimensional Vlasov-Poisson system*, Asymptotic Anal., **4**, 1991, pp. 187-214.

[8] I. Langmuir and K.T. Compton *Electrical Discharges in Gases: Part II, Fundamental Phenomena in Electrical Discharges*, Rev. Mod. Phys. **3** (1931), pp. 191-257.

[9] R.V. Lovelace, E. Ott, *Theory of magnetic insulation*, Phys. Fluids, **17**, No 6, 1974, pp. 1263-1268.

[10] F. Méhats, *The Child-Langmuir asymptotics applied to the bipolar diode*, to appear.

[11] F. Poupaud, *Boundary value problems for the stationary Vlasov-Maxwell system*, Forum Mathematicum, **4**, 1992, pp. 499-527.

[12] A. Ron, A.A. Mondelli, N. Rostoker, *Equilibria for magnetic insulation*, IEEE Trans. Plas. Sci., **PS-1**, No 4, 1973, pp. 85-93.