On singular perturbation problems for the nonlinear Poisson equation

A. Ambroso[1], F. Méhats[2], P.A. Raviart[1]

[1] CMAP (UMR 7641), École polytechnique, F-91128 Palaiseau
[2] MIP (UMR 5640) - UFR MIG, Université Paul Sabatier, 118 route de Narbonne, F-31062 Toulouse

Abstract

We study a singular perturbation problem for the nonlinear Poisson equation and we classify its solutions. This classification is based on their monotonicity properties, which mainly depend on the stationary points of the so-called *Sagdeev potential*. For each class of solution, we give necessary and sufficient conditions for the resolution of the problem and provide a boundary layer analysis.

Key words : Nonlinear Poisson equation, singular perturbation problem, Sagdeev potential, boundary layer.

Introduction

Consider the following singular perturbation problem for the nonlinear Poisson equation:

$$\left\{\begin{array}{l} \varepsilon^2 \dfrac{d^2\phi}{dx^2} = V'(\phi), \quad 0 < x < 1, \\[2ex] \phi(0) = \phi_0, \qquad \phi(1) = \phi_1, \end{array}\right. \tag{0.1}$$

where V is a given smooth function, usually referred as the *Sagdeev potential* of the system, and ε is a small parameter. Depending on the properties of V, this problem (0.1) may not have a unique solution: in this paper, we classify its different solutions and study their behavior as ε tends to 0.

Such problems occur for instance in plasma physics. Consider the transport of a system of non collisional charged particles (electrons and ions) in their self-consistent electric field between two perfectly conducting parallel plane electrodes. Such a problem can be modeled either by a Vlasov-Poisson or an Euler-Poisson system in one-space dimension. If we assume stationarity and we are given the electric potential $\phi = \phi(x)$, the stationary Vlasov or Euler equations describing the transport of charged particles can be solved to provide the total charge density of the plasma as a functional of ϕ denoted by $\mathcal{N}_\phi$. Then the self-consistent potential ϕ is characterized as the solution of the Poisson equation

$$-\frac{d^2\phi}{dx^2} = \frac{1}{\epsilon_0}\mathcal{N}_\phi.$$

In a number of cases, the functional $\mathcal{N}_\phi$ reduces to a function $\mathcal{N}(\phi)$. This is indeed the case of the Euler-Poisson system considered in Section 1. This is also true of the Vlasov-Poisson system analyzed in [2]. Now, passing in dimensionless variables (see [1]), we obtain a problem of the form (0.1) where ε stands for the ratio Debye length of the plasma over the electrode distance.

The aim of this paper is to provide a general analysis of the qualitative properties of the solutions of (0.1) based on very few assumptions on the Sagdeev potential V. In fact, the knowledge of the set of stationary points of V is enough to give the main behavior of ϕ and to provide an asymptotic analysis as $\varepsilon \to 0$. This fact is well known, we refer to [4] for a bibliography on this subject and more generally on these singular perturbations problems, which have been studied for a long time. These problems were also studied through non standard analysis in [3].

This paper is organized as follows. In Section 1, the Sagdeev potential is calculated in the case of the one-dimensional Euler-Poisson system. This case will be used as a reference example in the sequel of the paper, since it contains the different situations that we will meet. Each result of our paper will thus be illustrated by the Euler-Poisson model.

In Section 2, we introduce some notations and a classification of the solutions of (0.1), depending on their monotonicity properties. Section 3 is devoted to the study of monotone solutions: we prove that this system admits a solution for ε small enough if and only if V presents a stationary minimum; this solution is then unique. We also study the convergence of ϕ as $\varepsilon \to 0$ and analyze the boundary layers that appear in the domain.

In the case of non monotone solutions, the situation is much more complicated. Sections 4 and 5 are devoted to the solutions that present one or two changes of monotonicity. In Section 4, we introduce necessary and sufficient conditions for the existence of such non-monotone solutions, always for ε small enough. The convergence analysis as $\varepsilon \to 0$ is done in Section 5, which also includes a study of the boundary layers and of a free layer present inside the domain.

In Section 6, we state without proof extension of the above results to the more general solution (periodic).

1 An example: the Euler-Poisson Problem

Let us start with a fluid model of electrostatic sheaths which will play in all the sequel the role of reference example. We consider the flow of a plasma between two parallel planes: at one boundary a neutral plasma is injected; the other boundary acts as a pure absorbing electrode negatively biased with respect to the plasma which collects the ions and the electrons of the plasma. We suppose that the electrons behave as an isothermal fluid and the ions as an ideal polytropic fluid. In addition, we assume that the flow is parallel and stationary. Then, neglecting the inertia of electrons and *using dimensionless variables*, we find on the one hand that the electron density n_e is linked to the electric potential ϕ by the *Maxwell-Boltzmann* relation

$$n_e = \exp(-\phi) \ . \tag{1.1}$$

On the other hand, the density n and the mean velocity u of the ion fluid satisfy the density and momentum conservation equations

$$(1.2)\qquad \begin{cases} \dfrac{d}{dx}(nu) &= 0 \\[2ex] \dfrac{d}{dx}(nu^2 + \alpha n^\gamma) &= n\dfrac{d\phi}{dx} \end{cases}$$

where α is a positive constant and $\gamma > 1$ is the adiabatic exponent. Finally, the electron and ion densities are related to the electric potential by means of the Poisson equation

$$(1.3)\qquad \varepsilon^2 \frac{d^2\phi}{dx^2} = n - n_e$$

where $\varepsilon > 0$ is a small parameter. Indeed, ε is the ratio of the Debye length of the plasma at the injection to a characteristic length which can been chosen as the distance between the injection plane and the electrode.

We supplement the above equations with the boundary conditions

$$(1.4)\qquad \begin{cases} n(0) = 1 \quad , \quad u(0) = u_0 > 0 \\[2ex] \phi(0) = 0 \quad , \quad \phi(1) = \phi_c > 0 \ . \end{cases}$$

Using (1.2) and (1.4), a little algebra shows that

$$(1.5)\qquad \phi = F(n) \equiv \frac{u_0^2}{2}\left(\frac{1}{n^2} - 1\right) + \frac{\gamma\alpha}{\gamma - 1}(n^{\gamma-1} - 1) \ .$$

Therefore ϕ is a decreasing function of n for $n \in (0,\ n_L]$ and increasing for $n > n_L$. By an *a posteriori* argument, it can be shown that only the decreasing branch must be considered for our analysis. Indeed, for ε small enough, no solution can be such that the pair $(n(x_0), \phi(x_0))$ is on the increasing branch of the graph of F, for any $x_0 \in [0, 1]$. To prove this fact, one can use similar arguments as those of the proofs of Theorems 3.1 and 4.1.

Hence (1.5) enables us to define $n = N(\phi)$ on an interval $[\phi_L,\ +\infty)$ as a strictly monotone decreasing function of ϕ. Moreover, since by (1.4) $n(0)$ is prescribed, we must have $n(0) = 1 \in (0,\ n_L]$, thus $n_L \geq 1$. In fact, we have

$$(1.6)\qquad \phi_L = F(n_L) \ , \quad n_L = \left(\frac{u_0^2}{\gamma\alpha}\right)^{\frac{1}{\gamma+1}} \ .$$

Consequently n_L is greater than 1 as soon as the injection velocity u_0 satisfies the following condition which will be assumed throughout the paper:

$$(1.7)\qquad u_0 \geq \sqrt{\gamma\alpha} \ .$$

Finally the problem (1.1)-(1.4) amounts to find the electrostatic potential ϕ as a solution of the nonlinear Poisson problem

$$(1.8)\qquad \begin{cases} \varepsilon^2 \dfrac{d^2\phi}{dx^2} = V'(\phi) \ , \quad x \in (0,\ 1) \ , \\[2ex] \phi(0) = 0 \ , \qquad \phi(1) = \phi_c \ , \end{cases}$$

where

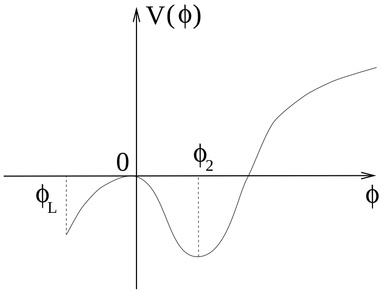
$$V(\phi) := \int_0^{\phi} (N(\psi) - \exp(-\psi))d\psi \tag{1.9}$$

is the so-called *Sagdeev potential.* The form of the graph of the function V depends strongly on the injection velocity. Indeed, defining the (scaled) ion acoustic velocity c_S by

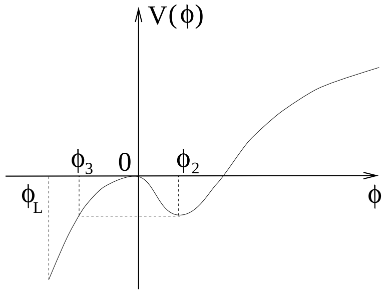
$$c_S = \sqrt{1 + \gamma\alpha}\ , \tag{1.10}$$

there exist two constants $\lambda < 1$ and $\mu > 1$ depending on α and γ such that the situations presented in the figures (1.a)-(1.e) hold. For a detailed analysis of the Sagdeev potential, we refer to [1].

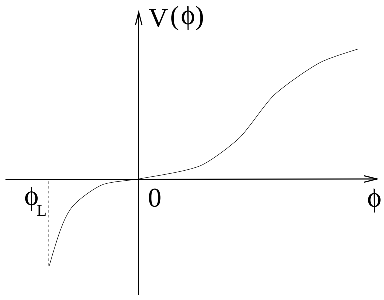
As a direct consequence of the hypothesis of neutrality of the plasma at the injection we have $V'(0) = N(0) - n_e(0) = 0$. Therefore, in each case, $\phi = 0$ is a stationary point for V. It can be shown that, when $u_0 \neq c_S$ and u_0 is not too large, there exists a $\phi_2 \neq 0$ such that again $V'(\phi_2) = 0$.



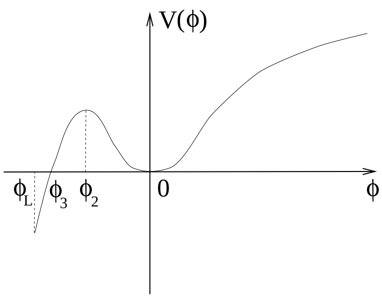
(1.a): $\sqrt{\gamma\alpha} < u_0 \leq \lambda\ c_S$



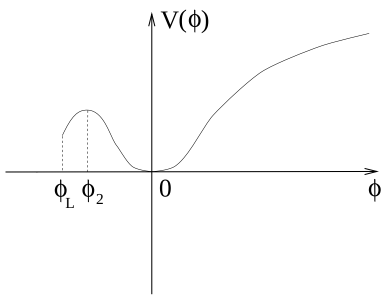
(1.b): $\lambda\ c_S < u_0 < c_S$



(1.c): $u_0 = c_S$



(1.d): $c_S < u_0 < \mu\ c_S$



(1.e): $u_0 > \mu\ c_S$

Fig. 1 : *Sagdeev Potential*

2 Notations and Preliminary Results

We go back to the general problem (0.1). We assume that V is a C^3 function from $[\phi_L,\ +\infty)$ into $\mathbb{R}$, such that

$$\lim_{\phi\to+\infty} V(\phi) = +\infty$$

and ε is a positive small parameter. Since $\psi(x) = \phi(1-x)$ satisfies the nonlinear Poisson equation with the boundary conditions

$$\psi(0) = \phi_1 \ , \qquad \psi(1) = \phi_0 \ ,$$

we can assume, without restricting the generality, that

$$\phi_1 \geq \phi_0 \ .$$

Also we can assume

$$V(\phi_0) = 0 \ .$$

We denote by Φ_s the set of the stationary points of V, *i.e.*,

$$\Phi_s := \{\psi \in [\phi_L,\ +\infty), \quad V'(\psi) = 0\} \ .$$

We set also

$$m(a,\ b) := \underset{\phi\in[a,b]}{\mathrm{Min}} \ V(\phi) \geq 0$$

and we suppose that V reaches its global minimum value in $[\phi_0,\ \phi_1]$ at N distinct points $(\phi_m^i)_{i=1,\dots,N}$, *i.e.*, for all i

$$V(\phi_m^i) = m(\phi_0,\ \phi_1) \ , \qquad \phi_0 \leq \phi_m^1 < \phi_m^2 < \dots < \phi_m^i < \dots \phi_m^N \leq \phi_1 \ .$$

Moreover, when ϕ is monotone increasing, we denote by x_ε^i $(0 \leq x_\varepsilon^i \leq 1)$ the unique point such that $\phi(x_\varepsilon^i) = \phi_m^i$.

All the analysis of this paper is based on the existence of a prime integral for the nonlinear Poisson equation (NLP). Indeed, multiplying this equation by $\dfrac{d\phi}{dx}$ and integrating it from 0 to x, we obtain

$$\left[\frac{d\phi}{dx}(x)\right]^2 = \delta^2 + \frac{2}{\varepsilon^2}V(\phi(x)) \ , \qquad \delta = \left|\frac{d\phi}{dx}(0)\right| \ . \tag{2.1}$$

We begin with a preliminary result

Lemma 2.1. *A solution ϕ of* (0.1) *satisfies one of the two exclusive properties*

(i) *ϕ has at most a local minimum on $[0,\ 1)$ ϕ_* at x_* and a local maximum on $(0,\ 1]$ ϕ^* at x^*, with $0 \leq x_* < x^* \leq 1$.*

(ii) *ϕ may be extended as a periodic function with period $T < 1$ and, in one period, it has at most a local minimum ϕ_* and a local maximum ϕ^*.*

Proof. First, we remark that, if ϕ is a solution of the NLP equation such that

$$\phi(x_2) = \phi(x_1), \qquad \frac{d\phi}{dx}(x_2) = \frac{d\phi}{dx}(x_1) \ ,$$

for some pair of points (x_1, x_2), $x_2 < x_1$, then the function ϕ may be extended as a periodic function with period $T = x_2 - x_1$. Now, let ϕ be a solution of (0.1). If there exist three points x_i, $1 \leq i \leq 3$, such that

$$\phi(x_1) = \phi(x_2) = \phi(x_3),$$

ϕ can be extended as a periodic function. Indeed, using (2.1), we find

$$\left|\frac{d\phi}{dx}(x_1)\right| = \left|\frac{d\phi}{dx}(x_2)\right| = \left|\frac{d\phi}{dx}(x_3)\right| \ .$$

Therefore, two among $\dfrac{d\phi}{dx}(x_i)$, $1 \leq i \leq 3$, coincide and we can apply the above remark.

Let ϕ be a non-periodic solution, *i.e.*, there does not exists three points x_i, $1 \leq i \leq 3$, such that

$$\phi(x_1) = \phi(x_2) = \phi(x_3) \ .$$

If $x_* \in [0,\ 1)$ is a minimum point of ϕ, *i.e.*,

$$\phi_* = \phi(x_*) = \operatorname*{Min}_{x \in [0,\ 1]} \ \phi(x) \ ,$$

there cannot exist another local minimum at a point $x_0 \in [0,\ 1)$. Indeed, we must have $\phi(x_0) \geq \phi_*$ and, for $\psi > \phi(x_0)$, $\psi - \phi(x_0)$ small enough, there exist at least three points x_j such that $\phi(x_j) = \psi$, which is excluded. The same argument holds to show that there exists at most one local maximum at $x^* \in (0,\ 1]$. Finally, we have $x_* < x^*$; otherwise, since $\phi_* \leq \phi_0 < \phi_1 \leq \phi^*$, there would exist at least three points x_j, such that, for all $\psi \in (\phi_0,\ \phi_1)$

$$\phi(x_j) = \psi$$

In the same way, we consider a solution ϕ that can be extended as a periodic function with period $T < 1$. Again if $x_* \in [0,\ T)$ is a minimum point of ϕ, we obtain by arguing as above that ϕ cannot have another local maximum in $[0,\ T)$. Similarly, one proves that ϕ has at most a local maximum in $[0,\ T)$.

■

As a consequence of Lemma 2.1, a solution of the Problem (0.1) must belong to one of the five categories below:

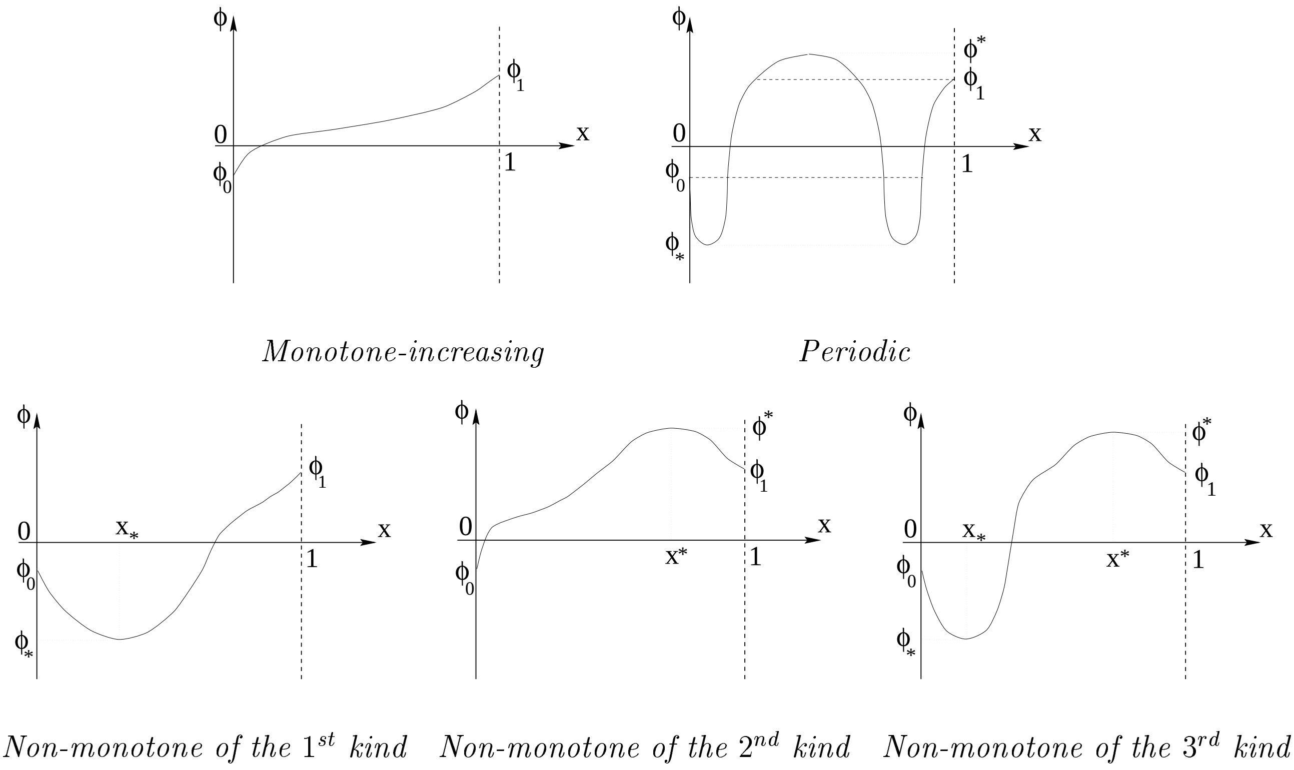
(i) Monotone-increasing,

(ii) Periodic,

(iii) Non-monotone of the 1*st kind :* ϕ is decreasing in $[0,\ x_*]$ then increasing in $[x_*,\ 1]$,

(iv) Non-monotone of the 2*nd kind :* ϕ is increasing in $[0,\ x^*]$ then decreasing in $[x^*,\ 1]$,

(v) Non-monotone of the 3*rd kind :* ϕ is decreasing in $[0,\ x_*]$ then increasing in $[x_*,\ x^*]$ and again decreasing in $[x^*,\ 1]$.



Monotone-increasing *Periodic*

Non-monotone of the 1^{st} *kind* *Non-monotone of the* 2^{nd} *kind* *Non-monotone of the* 3^{rd} *kind*

Fig. 2 : *Examples of solutions*

3 Monotone Solutions

3.1 Existence at Fixed ε and Convergence Results

We begin by studying the simple case of monotone increasing solutions of the nonlinear Poisson problem (0.1). The analysis of this case will give us some hints for the sequel and it will emphasize the fundamental role played by the stationary minimum points of the function V.

We can state

Theorem 3.1. *Assume that $\phi_m^i \in \Phi_s$ for at least one i. The problem* (0.1) *has a monotone-increasing solution ϕ_ε for all $\varepsilon > 0$. On the other hand, if V does not present a stationary minimum in $[\phi_0,\ \phi_1]$ (i.e., $\phi_m^i \notin \Phi_s$ for all ϕ_m^i, $1 \le i \le N$), the problem* (0.1) *has a monotone-increasing solution only for ε large enough. In any case, this solution is unique in the class of monotone solutions.*

Proof. We are looking for a monotone-increasing solution of (0.1). By (2.1), we have

$$\frac{d\phi}{dx} = \sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\phi)}, \quad \delta = \frac{d\phi}{dx}(0) \ge 0.$$

Using the boundary condition $\phi(0) = \phi_0$, we find

$$\int_{\phi_0}^{\phi(x)} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} = x. \tag{3.1}$$

It remains only to find $\delta \geq 0$ so that the boundary condition $\phi(1) = \phi_1$ holds. Hence setting

$$f(\delta) := \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2} V(\psi)}}, \tag{3.2}$$

looking for a monotone-increasing solution of (0.1) amounts to find $\delta \geq 0$ solution of the equation

$$f(\delta) = 1. \tag{3.3}$$

In the following this solution will be noted δ_ε.

Since $V(\phi_0) = 0$, we have $m(\phi_0,\ \phi_1) \leq 0$, and we can set

$$\delta_1 := \frac{1}{\varepsilon}\sqrt{-2m(\phi_0,\ \phi_1)}\ . \tag{3.4}$$

We observe that the function f is defined on the interval $(\delta_1,\ +\infty)$ and is clearly monotone-decreasing. In addition, we have

$$\lim_{\delta \to +\infty} f(\delta) = 0 \quad \text{and} \quad \lim_{\delta \to \delta_1} f(\delta) = \frac{\varepsilon}{\sqrt{2}} \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{V(\psi) - m(\phi_0,\ \phi_1)}}.$$

If $\phi_m^i \in \Phi_s$ for at least one i, we can write

$$V(\psi) - m(\phi_0,\ \phi_1) = V(\psi) - V(\phi_m^i) = O((\psi - \phi_m^i)^2), \quad \psi \to \phi_m^i$$

and the above integral diverges so that

$$\lim_{\delta \to \delta_1} f(\delta) = +\infty.$$

Hence, in that case, Eq.(3.3) has a unique solution $\delta_2 > \delta_1$ or equivalently, for all $\varepsilon > 0$, the problem (0.1) has a unique monotone-increasing solution given by (3.1).

On the other hand, if $\phi_m^i \notin \Phi_s$ for all i, the above integral is convergent and Eq. (3.3) has a unique solution $\delta \geq \delta_1$ if and only if

$$f\left(\frac{1}{\varepsilon}\sqrt{-2m(\phi_0,\ \phi_1)}\right) \geq 1 \Longleftrightarrow \frac{\varepsilon}{\sqrt{2}} \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{V(\psi) - m(\phi_0,\ \phi_1)}} \geq 1$$

i.e., if and only if ε is large enough ! ∎

As ε is in fact a small parameter, the existence of the solution ϕ_ε for ε large enough is of little interest.

We remember that x_ε^i is the unique point such that $\phi(x_\varepsilon^i) = \phi_m^i$. By using (3.1) we have

$$x_\varepsilon^i = \varepsilon \int_{\phi_0}^{\phi_m^i} \frac{d\psi}{\sqrt{\delta^2 \varepsilon^2 + 2V(\psi)}}\ .$$

We next state two technical lemmas that give the asymptotic behavior of this integral when $\phi_m^i \in \Phi_s$ and in the sequel we use these lemmas to study the convergence of the points x_ε^i for

$\varepsilon \to 0$.

Let W be a $\mathcal{C}^3$ function defined in an interval $[\phi_l,\ \phi_r]$ which possesses a *unique* minimum ϕ_m and assume one of the two exclusive properties

$$W'(\phi_m) = 0\ , \qquad W''(\phi_m) = k > 0 \tag{3.5}$$

$$W'(\phi_m) = 0\ , \qquad W''(\phi_m) = 0 \quad W'''(\phi_m) = \theta > 0 \tag{3.6}$$

For simplicity we assume $W(\phi_m) = 0$.

Lemma 3.2. *Assume the hypotheses* (3.5)*. Then for all fixed* $\phi \in [\phi_l,\ \phi_r]$*, we have*

$$\left| \int_{\phi_m}^{\phi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} \right| \sim -\frac{1}{\sqrt{k}} \operatorname{Log} \alpha\ , \tag{3.7}$$

as $\alpha \to 0$.

Lemma 3.3. *Assume the hypotheses* (3.6)*. Then for all fixed* $\phi \in [\phi_m,\ \phi_r]$*, we have*

$$\int_{\phi_m}^{\phi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} \sim C \left(\frac{3}{\theta\alpha}\right)^{1/3}\ ,$$

where C *is a positive constant.*

For the sake of completeness we give the proofs of these lemmas in Appendix.

Obviously one can consider the general case where $W'(\phi_m) = W''(\phi_m) = \cdots = W^k(\phi_m) = 0$, $W^{k+1}(\phi_m) > 0$. We limit our analysis to the cases (3.5) and (3.6) since they are the ones that naturally appear in the study of the Euler-Poisson problem.

3.2 Convergence Results. Case of a Simple Stationary Point

We begin by considering the case of a *simple stationary point*: there exists a unique point $\phi_m \in [\phi_0,\ \phi_1] \cap \Phi_s$ that minimizes V on $[\phi_0,\ \phi_1]$. We first prove

Lemma 3.4. *Assume the following hypotheses:*

(3.8) *there exists a unique point* $\phi_m \in [\phi_0,\ \phi_1] \cap \Phi_s$ *that minimizes* V*,*

(3.9) $V''(\phi_m) > 0$,

(3.10) $\phi_m \neq \phi_0$ *and* $\phi_m \neq \phi_1$.

Then we obtain

$$\lim_{\varepsilon \to 0} x_\varepsilon = \frac{1}{2} \tag{3.11}$$

We remark that if $\phi_m = \phi_0$ (resp. $\phi_m = \phi_1$), we have directly $x_\varepsilon = 0$ (resp. $x_\varepsilon = 1$).

Proof. We write

$$x_\varepsilon = \varepsilon \int_{\phi_0}^{\phi_m} \frac{d\psi}{\sqrt{\varepsilon^2(\delta_\varepsilon^2 - \delta_1^2) + 2(V(\psi) - V(\phi_m))}} \tag{3.12}$$

and

$$1 - x_\varepsilon = \varepsilon \int_{\phi_m}^{\phi_1} \frac{d\psi}{\sqrt{\varepsilon^2(\delta_\varepsilon^2 - \delta_1^2) + 2(V(\psi) - V(\phi_m))}} \ . \tag{3.13}$$

In order to apply Lemma 3.2 to the above integrals with

$$\alpha^2 = (\delta_\varepsilon^2 - \delta_1^2)\varepsilon^2 \quad \text{and} \quad W(\phi) = 2(V(\phi) - V(\phi_m)) \ ,$$

we need first to show that $\alpha \to 0$ as $\varepsilon \to 0$. Using the prime integral (2.1) and (3.4), we have

$$(\phi'(x_\varepsilon))^2 = \delta_\varepsilon^2 + \frac{2}{\varepsilon^2} V(\phi_m) = \delta_\varepsilon^2 - \delta_1^2 > 0 \ .$$

Since

$$V(\phi_m) \leq V(\phi) \quad \forall \phi \in [\phi_0, \ \phi_1] \ ,$$

we deduce again from (2.1) that $\phi'(x)$ reaches its minimum at $x = x_\varepsilon$. Then, by Rolle's Theorem, there exists a point $\bar{x} \in (0,1)$ such that $\phi'(\bar{x}) = \phi_1 - \phi_0$ and we have

$$\delta_\varepsilon^2 - \delta_1^2 = (\phi'(x_\varepsilon))^2 \leq (\phi'(\bar{x}))^2 = (\phi_1 - \phi_0)^2$$

which proves that

$$\alpha^2 = (\delta_\varepsilon^2 - \delta_1^2)\varepsilon^2 \to 0 \quad \text{for} \quad \varepsilon \to 0 \ .$$

Setting $k = V''(\phi_m)$, we apply Lemma 3.2 to the integrals in (3.12) and (3.13), which yields

$$\frac{x_\varepsilon}{\varepsilon} \ \sim \ -\frac{1}{\sqrt{k}} \operatorname{Log}(\varepsilon \sqrt{\delta_\varepsilon^2 - \delta_1^2}) \ , \tag{3.14}$$

$$\frac{1 - x_\varepsilon}{\varepsilon} \ \sim \ -\frac{1}{\sqrt{k}} \operatorname{Log}(\varepsilon \sqrt{\delta_\varepsilon^2 - \delta_1^2}) \ . \tag{3.15}$$

The sequence $\{x_\varepsilon\}$ being bounded, we can extract a subsequence converging towards a value x_0. It follows from (3.14) and (3.15) that $x_0 = 1/2$. Thus the limit of all subsequences is unique and we can conclude. ∎

If (3.10) is not verified, then (3.11) is obviously replaced by $x_\varepsilon \equiv 0$ in the case $\phi_m = \phi_0$ or $x_\varepsilon \equiv 1$ in the case $\phi_m = \phi_1$. Moreover, we can remark that the situation corresponding to $V''(\phi_m) = 0, \ V'''(\phi_m) \neq 0$ is only possible in the following two cases

(i) $\phi_m = \phi_0$ and $V'''(\phi_0) > 0$,

(ii) $\phi_m = \phi_1$ and $V'''(\phi_1) < 0$,

which also leads to $x_\varepsilon \equiv 0$ or $x_\varepsilon \equiv 1$.

For the monotone solution ϕ_ε of (0.1), we have the following convergence result:

Theorem 3.5. *Assume that there is a unique point $\phi_m \in [\phi_0, \phi_1] \cap \Phi_s$ that minimizes V. Then, as $\varepsilon \to 0$, we have for all $a > 0$ arbitrary small*

$$
\left\{
\begin{array}{llll}
\phi_\varepsilon \to \phi_m & \textit{uniformly on } [a, 1-a] & \textit{if} & \phi_m \in (\phi_0, \phi_1), \\
\phi_\varepsilon \to \phi_0 & \textit{uniformly on } [0, 1-a] & \textit{if} & \phi_m = \phi_0, \\
\phi_\varepsilon \to \phi_1 & \textit{uniformly on } [a, 1] & \textit{if} & \phi_m = \phi_1.
\end{array}
\right.
\tag{3.16}
$$

Proof. For specificity, we consider the (most complicated) case where $\phi_m \in (\phi_0, \phi_1)$. Let $x \in (0, 1)$; we want to check that

$$
\phi_\varepsilon(x) \to \phi_m \quad \text{as } \varepsilon \to 0 \tag{3.17}
$$

from which the conclusion follows by the monotonicity of the function ϕ_ε. The proof is by contradiction. If (3.17) does not hold, we may assume that, at least for a subsequence still denoted by (ϕ_ε), we have

$$
\phi_\varepsilon(x) \to \chi \neq \phi_m.
$$

We first consider the case where $\chi < \phi_m$. As $\delta_\varepsilon > \delta_1$, Eq. (3.1) yields

$$
x = \varepsilon \int_{\phi_0}^{\phi_\varepsilon(x)} \frac{d\psi}{\sqrt{\delta_\varepsilon^2 \varepsilon^2 + 2V(\psi)}} < \frac{\varepsilon}{\sqrt{2}} \int_{\phi_0}^{\phi_\varepsilon(x)} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_m)}}.
$$

Since ϕ_m is the unique stationary point that minimizes V, we find

$$
\int_{\phi_0}^{\phi_\varepsilon(x)} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_m)}} \to \int_{\phi_0}^{\chi} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_m)}} < +\infty \ .
$$

Therefore we obtain $x = 0$ by passing to the limit, which is impossible. On the other hand, if $\chi > \phi_m$, we write from (3.1)

$$
1 - x = \varepsilon \int_{\phi_\varepsilon(x)}^{\phi_1} \frac{d\psi}{\sqrt{\delta_\varepsilon^2 \varepsilon^2 + 2V(\psi)}}
$$

which yields

$$
1 - x < \frac{\varepsilon}{\sqrt{2}} \int_{\phi_\varepsilon(x)}^{\phi_1} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_m)}}.
$$

Since here

$$
\int_{\phi_\varepsilon(x)}^{\phi_1} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_m)}} \to \int_{\chi}^{\phi_1} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_m)}} < +\infty,
$$

we obtain $x = 1$, which leads to a contradiction with $x \in]0, 1[$. Hence every converging subsequence of ϕ_ε converges to ϕ_m which proves that the whole sequence converges to ϕ_m.

When $\phi_m = \phi_0$, one proves in exactly the same way that $\phi_\varepsilon(x) \to \phi_0$ for all $x \in [0, 1)$ while, when $\phi_m = \phi_1$, one obtains that $\phi_\varepsilon(x) \to \phi_1$ for all $x \in (0, 1]$. The conclusion (3.16) follows. ∎

3.3 Convergence Result. Case of Multiple Stationary Points

We pass to the case of *multiple stationary points* where

$$\text{(3.18)} \qquad \operatorname*{Min}_{\phi\in[\phi_0,\ \phi_1]} \quad V(\phi) = V(\phi_m^i) \qquad \text{and} \qquad V'(\phi_m^i) = 0, \quad 1 \le i \le N \ .$$

$$\text{(3.19)} \qquad k_i := V''(\phi_m^i) > 0 \qquad \text{for all} \quad i \ .$$

We recall that x_ε^i, already defined in Section 3.1, is the unique point such that $\phi_\varepsilon(x_\varepsilon^i) = \phi_m^i$. Again we want to use Lemma 3.2 to study the convergence of δ_ε and the points x_ε^i. We denote by $\Delta(a,\ b)$ the Kronecker symbol (which vanishes everywhere except for $a = b$).

Lemma 3.6. *Assume the hypotheses* (3.18) *and* (3.19)*. Then we have for all* i

$$x_\varepsilon^i \to \frac{\sum_{j=1}^{i-1} \frac{2}{\sqrt{k_j}} + \frac{1}{\sqrt{k_i}} - \Delta(\phi_0,\ \phi_m^1)\frac{1}{\sqrt{k_1}}}{\sum_{j=1}^{N} \frac{2}{\sqrt{k_j}} - \Delta(\phi_0,\ \phi_m^1)\frac{1}{\sqrt{k_1}} - \Delta(\phi_m^N,\ \phi_1)\frac{1}{\sqrt{k_N}}} \quad \textit{as} \quad \varepsilon \to 0 \ .$$

Proof. We have
(3.20)

$$\begin{aligned} x_\varepsilon^i &= \varepsilon \int_{\phi_0}^{\phi_m^i} \frac{d\psi}{\sqrt{\varepsilon^2\delta_\varepsilon^2 + 2V(\psi)}} \\ &= \varepsilon \left\{ \int_{\phi_0}^{\phi_m^1} + \int_{\phi_m^1}^{\frac{\phi_m^1+\phi_m^2}{2}} + \int_{\frac{\phi_m^1+\phi_m^2}{2}}^{\phi_m^2} + \cdots + \int_{\frac{\phi_m^{i-1}+\phi_m^i}{2}}^{\phi_m^i} \right\} \frac{d\psi}{\sqrt{\varepsilon^2\delta_\varepsilon^2 + 2V(\psi)}} \\ &= \varepsilon \left\{ \int_{\phi_0}^{\phi_m^1} + \int_{\phi_m^1}^{\frac{\phi_m^1+\phi_m^2}{2}} + \int_{\frac{\phi_m^1+\phi_m^2}{2}}^{\phi_m^2} + \cdots + \int_{\frac{\phi_m^{i-1}+\phi_m^i}{2}}^{\phi_m^i} \right\} \frac{d\psi}{\sqrt{\varepsilon^2(\delta_\varepsilon^2 - \delta_1^2) + 2(V(\psi) - V(\phi_m^i))}} \end{aligned}$$

and
(3.21)

$$\begin{aligned} 1 - x_\varepsilon^i &= \varepsilon \int_{\phi_m^i}^{\phi_1} \frac{d\psi}{\sqrt{\varepsilon^2\delta_\varepsilon^2 + 2V(\psi)}} \\ &= \varepsilon \left\{ \int_{\phi_m^i}^{\frac{\phi_m^i+\phi_m^{i+1}}{2}} + \cdots + \int_{\frac{\phi_m^{N-1}+\phi_m^N}{2}}^{\phi_m^N} + \int_{\phi_m^N}^{\phi_1} \right\} \frac{d\psi}{\sqrt{\varepsilon^2(\delta_\varepsilon^2 - \delta_1^2) + 2(V(\psi) - V(\phi_m^i))}} \ . \end{aligned}$$

We set

$$\alpha = \varepsilon\sqrt{\delta_\varepsilon^2 - \delta_1^2} \quad \text{and} \quad W(\phi) = 2(V(\phi) - V(\phi_m^i)) \ .$$

Exactly as in Lemma 3.8, we can show that $\alpha \to 0$ for $\varepsilon \to 0$. Then we can apply Lemma 3.2 to the integrals above and we obtain

$$\text{(3.22)} \qquad \frac{x_\varepsilon^i}{\varepsilon} \sim -\left\{ \sum_{j=1}^{i-1} \frac{2}{\sqrt{k_j}} + \frac{1}{\sqrt{k_i}} - \Delta(\phi_0,\ \phi_m^1)\frac{1}{\sqrt{k_1}} \right\} \mathrm{Log}(\sqrt{\delta_\varepsilon^2 - \delta_1^2}\,\varepsilon) \ ,$$

$$(3.23) \qquad \frac{1-x_\varepsilon^i}{\varepsilon} \;\sim\; -\left\{\frac{1}{\sqrt{k_i}} + \sum_{j=i+1}^{N} \frac{2}{\sqrt{k_j}} - \Delta(\phi_m^N,\ \phi_1)\frac{1}{\sqrt{k_N}}\right\} \mathrm{Log}(\sqrt{\delta_\varepsilon^2 - \delta_1^2\varepsilon}) \ .$$

The sequence $\{x_\varepsilon^i\}$ being bounded, we can extract a subsequence converging towards a value x_0^i. Using (3.22) and (3.23), we obtain

$$x_0^i = \frac{\sum_{j=1}^{i-1} \frac{2}{\sqrt{k_j}} - \Delta(\phi_0,\ \phi_m^1)\frac{1}{\sqrt{k_1}} + \frac{1}{\sqrt{k_i}}}{\sum_{j=1}^{N} \frac{2}{\sqrt{k_j}} - \Delta(\phi_0,\ \phi_m^1)\frac{1}{\sqrt{k_1}} - \Delta(\phi_m^N,\ \phi_1)\frac{1}{\sqrt{k_N}}} \ .$$

Thus the limit of all subsequences is unique and we can conclude

$$x_\varepsilon^i \to x_0^i \quad \text{for} \quad \varepsilon \to 0 \ .$$

■

Now, for all i, we consider a point in $(\phi_i,\ \phi_{i+1})$. Let us choose arbitrary the middle points $\dfrac{\phi_m^i + \phi_m^{i+1}}{2}$. For all i, we call $x_\varepsilon^{i,i+1}$ the point such that

$$\phi(x_\varepsilon^{i,i+1}) = \frac{\phi_m^i + \phi_m^{i+1}}{2} \ .$$

Such a point is unique by the monotonicity of ϕ. In the same way than above we can show that, when $\varepsilon \to 0$

$$x_\varepsilon^{i,i+1} \to x_0^{i,i+1} = \frac{\sum_{j=1}^{i} \frac{2}{\sqrt{k_j}} - \Delta(\phi_0,\ \phi_m^1)\frac{1}{\sqrt{k_1}}}{\sum_{j=1}^{N} \frac{2}{\sqrt{k_j}} - \Delta(\phi_0,\ \phi_m^1)\frac{1}{\sqrt{k_1}} - \Delta(\phi_m^N,\ \phi_1)\frac{1}{\sqrt{k_N}}} \ .$$

For the monotone solution of (0.1), we have the analogous result as Theorem 3.5:

Theorem 3.7. *Assume the hypotheses* (3.18) *and* (3.19). *Then, as* $\varepsilon \to 0$, *we have for all* $a > 0$ *arbitrary small*

$$\phi_\varepsilon \;\to\; \phi_m^1 \quad \textit{unif.} \quad \begin{cases} \textit{on} \quad [a,\ x_0^{1,2} - a]\ , & \textit{if} \quad \phi_m^1 > \phi_0 \\ \textit{on} \quad [0,\ x_0^{1,2} - a]\ , & \textit{if} \quad \phi_m^1 = \phi_0 \end{cases}$$

$$\phi_\varepsilon \;\to\; \phi_m^i \quad \textit{unif.} \quad \textit{on} \quad [x_0^{i-1,i} + a,\ x_0^{i,i+1} - a] \quad i = 2, \ldots, N-1$$

$$\phi_\varepsilon \;\to\; \phi_m^N \quad \textit{unif.} \quad \begin{cases} \textit{on} \quad [x_0^{N-1,N} + a,\ 1 - a]\ , & \textit{if} \quad \phi_m^N < \phi_1 \\ \textit{on} \quad [x_0^{N-1,N} + a,\ 1 - a]\ , & \textit{if} \quad \phi_m^N = \phi_1 \end{cases}$$

An example of monotone solution of (0.1) for small ε is shown in Fig. 3, where we consider a situation in which there are two points $\phi_m^1 < \phi_m^2$ in $(\phi_0,\ \phi_1) \cap \Phi_s$ that minimize V.

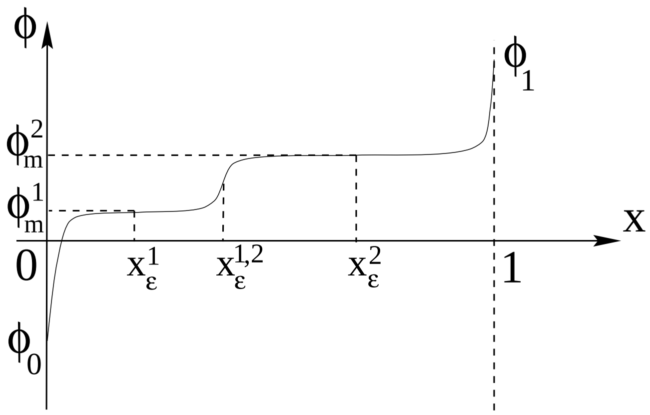


Fig. 3 : *A monotone solution at small* ε

Example: The Euler-Poisson Problem (continued)

In the case of Problem (1.8) we have

$$\begin{cases} \Phi_s = \{0,\ \phi_2 > 0\}, \quad m(0,\ \phi_c) = \begin{cases} \phi_2 & \text{if} \quad \phi_2 < \phi_c \\ \phi_c & \text{if} \quad \phi_2 > \phi_c \end{cases} & \text{if } u_0 < c_S \\ \\ \Phi_s = \{0\}, \qquad\qquad\quad m(0,\ \phi_c) = 0, & \text{if } u_0 = c_S,\ u_0 > \mu\ c_S \\ \\ \Phi_s = \{\phi_2 < 0,\ 0\}, \quad m(0,\ \phi_c) = 0, & \text{if } c_S < u_0 \le \mu\ c_S. \end{cases}$$

Therefore applying Theorem 3.1 we obtain that, for $u_0 \ge c_S$, Problem (1.8) admits a unique monotone increasing solution for all $\varepsilon > 0$, while, if $u_0 < c_S$ it does only if $\phi_c \ge \phi_2$.
When $u_0 < c_S$ and $\phi_c < \phi_2$, Problem (1.8) admits an increasing solution only for ε large enough.

In the limit $\varepsilon \to 0$, the behavior of monotone solutions of the Euler-Poisson problem is given by Theorem 3.5. We have

$$\text{(3.24)} \qquad \begin{cases} \phi_\varepsilon \to 0 & \text{uniformly on} & [0, 1-a] & \text{if} & & u_0 \ge c_S\ , \\ \\ \phi_\varepsilon \to \phi_2 & \text{uniformly on} & [a, 1-a] & \text{if} & u_0 < c_S \quad \text{and} & \phi_c > \phi_2\ , \\ \\ \phi_\varepsilon \to \phi_2 & \text{uniformly on} & [a, 1] & \text{if} & u_0 < c_S \quad \text{and} & \phi_c = \phi_2\ . \end{cases}$$

Therefore, if $u_0 \ge c_S$, the monotone solution presents a boundary layer at $x = 1$. If $u_0 < c_S$, it presents a boundary layer at $x = 0$ and also one at $x = 1$, in the case $\phi_c > \phi_2$.

In fact, the injection boundary $x = 0$ is to be considered as an *artificial boundary* inside a quasineutral plasma. Hence a boundary layer at $x = 0$ appears to be clearly unphysical. In realistic cases we must suppose that the so-called *Bohm's condition*

$$u_0 \ge c_S$$

is satisfied (see [5] for a review on this subject). If this condition holds, for small ε we have $\phi \simeq 0$ except in a boundary layer of width $\mathcal{O}(\varepsilon)$ located at $x = 1$. This boundary layer -which is analyzed in Subsection 3.4- is called an *anode sheath* in the plasma physics terminology.

■

3.4 Boundary Layer Analysis

In the previous section we studied the convergence for $\varepsilon \to 0$ of the monotone solution ϕ_ε of Problem 0.1 in the interior of the domain $[0,\ 1]$. Theorem 3.5 and Theorem 3.7 give results of uniform convergence on closed sets that in general do not contain the boundary. We want now to better analyze the convergence in a neighborhood of the boundary points. For the asymptotic analysis we restrict our study to the case of a simple stationary point. The case of multiple stationary points can be analyzed in an analogous way.

Lemma 3.8. *Assume the hypotheses* (3.8) *and* (3.9) *Then we have for some constant* $C > 0$ *independent of* ε

$$0 \leq \delta_\varepsilon - \delta_1 \leq \frac{1}{\varepsilon}\ \exp(-\frac{C}{\varepsilon}) \tag{3.25}$$

where δ_1 *is defined as in* (3.4) *by* $\delta_1 = \frac{1}{\varepsilon}\sqrt{-2m(\phi_0, \phi_1)}$.

Proof. We only consider the most difficult case where $\phi_m \in (\phi_0,\ \phi_1)$. Lemma 3.4 yields

$$x_\varepsilon \to \frac{1}{2} \quad \text{for} \quad \varepsilon \to 0\ .$$

By (3.14), for ε small enough, we obtain

$$-\frac{\varepsilon}{\sqrt{k}}\,\mathrm{Log}(\varepsilon\sqrt{\delta_\varepsilon^2 - \delta_1^2}) \geq \frac{1}{3}$$

which implies the estimate

$$\delta_\varepsilon^2 - \delta_1^2 \leq \frac{1}{\varepsilon^2}\exp\left(-\frac{2}{3}\frac{\sqrt{k}}{\varepsilon}\right)\ .$$

Since $\phi_m > \phi_0$, we know that $m(\phi_0, \phi_1) > 0$ and we can write

$$\delta_\varepsilon - \delta_1 \leq \frac{\delta_\varepsilon^2 - \delta_1^2}{\delta_\varepsilon + \delta_1} \leq \frac{1}{\varepsilon^2}\exp\left(-\frac{2}{3}\frac{\sqrt{k}}{\varepsilon}\right)\ \frac{\varepsilon}{\sqrt{-2m(\phi_0, \phi_1)}}\ .$$

(3.25) is obtained directly.

∎

It is easy to prove the analogue of Lemma 3.8 when $V''(\phi_m) = 0$.

Lemma 3.9. *Assume the hypotheses* (3.8) *and*

$$V''(\phi_m) = 0,\ V'''(\phi_m) \neq 0\ . \tag{3.26}$$

Then we have for some constant $C > 0$ *independent of* ε

$$0 \leq \delta_\varepsilon - \delta_1 \leq C\ \varepsilon^2\ .$$

In general a monotone solution of (0.1) ϕ_ε can present a boundary layer at both points $x = 0$ and $x = 1$. The boundary layer at $x = 0$ exists if and only if $\phi_m > \phi_0$, while the boundary layer at $x = 1$ exists if and only if $\phi_m < \phi_1$. The analysis of these boundary layers can be done by using similar techniques as those developed in [2]. For the reader's convenience we recall the main steps of the study of the boundary layer at $x = 0$. Introduce the change of scale

$$x = \varepsilon y$$

and set

$$\chi_\varepsilon(y) = \phi_\varepsilon(\varepsilon y). \tag{3.27}$$

The function χ_ε is solution of

$$\begin{cases} \dfrac{d^2\chi_\varepsilon}{dy^2} = V'(\chi_\varepsilon)\ , & 0 < y < \dfrac{x_\varepsilon}{\varepsilon}, \\ \chi_\varepsilon(0) = \phi_0, \qquad \chi_\varepsilon(\dfrac{x_\varepsilon}{\varepsilon}) = \phi_m\ . \end{cases} \tag{3.28}$$

We associate to this problem its formal limit

$$\begin{cases} \dfrac{d^2\chi_0}{dy^2} = V'(\chi_0)\ , & y > 0, \\ \chi_0(0) = \phi_0\ , \qquad \lim\limits_{y\to+\infty} \chi_0(y) = \phi_m\ . \end{cases} \tag{3.29}$$

The boundary layer analysis consists of three steps. We start by solving the formal limit problem (3.29)

Theorem 3.10. *Under the hypothesis* (3.8)*, the problem* (3.29) *has a monotone-increasing solution* χ_0*. Such a solution is unique.*

Then we show that $\chi_0(y)$, as $y \to +\infty$, converges to the stationary point ϕ_m "exponentially" or "quadratically", depending on the behavior of the function V in the neighborhood of this point. More precisely, we can state

Lemma 3.11. *Assume the hypotheses* (3.8) *and* (3.9)*. There exist two positive constants* $C_1,\ C_2$*, such that*

$$0 \le \phi_m - \chi_0(y) \le C_2 \exp(-C_1 y) \qquad \forall y \ge 0\ . \tag{3.30}$$

Lemma 3.12. *Assume the hypotheses* (3.8) *and* (3.26)*. There exist three positive constants* $C,\ C_1,\ C_2$ *such that*

$$0 \le \phi_m - \chi_0(y) \le \frac{1}{[C_1 + C_2(y - C)]^2} \qquad \forall y \ge C\ . \tag{3.31}$$

In the following we will say that the boundary layer is *exponential* (resp. *parabolic*) if Lemma 3.11 (resp. Lemma 3.12) holds.

Finally we derive an estimate of the error $\chi_\varepsilon - \chi_0$ obtained when approximating the solution χ_ε of (3.28) by the solution χ_0 of the corresponding formal limit problem (3.29).

Theorem 3.13. *Assume the hypothesis* (3.8)*. There exists a constant* $C > 0$ *independent of* ε *such that for all* $y \in [0, \frac{x_\varepsilon}{\varepsilon}]$

$$|\chi_\varepsilon(y) - \chi_0(y)| \leq \exp(-\frac{C}{\varepsilon}) \qquad \textit{if (3.9) holds,} \tag{3.32}$$

$$|\chi_\varepsilon(y) - \chi_0(y)| \leq C\ \varepsilon^2 \qquad \textit{if (3.26) holds.} \tag{3.33}$$

We can now give the proofs of the above statements.

Proof of Theorem 3.10 . The proof is similar to that of Theorem 3.1. Setting

$$\alpha_0 = \frac{d\chi_0}{dy}(0) \ ,$$

straightforward calculations show that looking for a monotone-increasing solution of (3.29) amounts to find $\alpha_0 > 0$ verifying

$$\lim_{\chi_0 \to \phi_m} \int_{\phi_0}^{\chi_0} \frac{d\psi}{\sqrt{\alpha_0^2 + 2V(\psi)}} = +\infty \ .$$

As ϕ_m is the unique minimizer of V and since we have

$$\text{either (hypothesis (3.9))} \quad V(\psi) - V(\phi_m) = O((\phi_m - \psi)^2) \quad \text{as} \quad \psi \to \phi_m,$$

$$\text{or (hypothesis (3.26))} \quad V(\psi) - V(\phi_m) = O((\phi_m - \psi)^3) \quad \text{as} \quad \psi \to \phi_m,$$

the integral

$$\int_{\phi_0}^{\phi_m} \frac{d\psi}{\sqrt{2(V(\psi) - V(\phi_m))}}$$

is well defined and diverges. Therefore the only convenient value for α_0 is

$$\alpha_0 = \sqrt{-2V(\phi_m)} \tag{3.34}$$

and the unique monotone-increasing solution χ_0 of (3.29) is given by

$$\int_{\phi_0}^{\chi_0(y)} \frac{d\psi}{\sqrt{2(V(\psi) - V(\phi_m))}} = y \ . \tag{3.35}$$

∎

Proof of Lemma 3.11 . The lower bound is obvious. Let us show the upper one. We know that, given $\alpha \in (0,\ k = V''(\phi_m))$, there exists $\eta \in [\phi_0,\ \phi_m)$ such that

$$V(\psi) - V(\phi_m) \geq \frac{k - \alpha}{2}(\phi_m - \psi)^2 \quad \text{for} \quad \psi \in [\eta,\ \phi_m] \ .$$

As $\chi_0(y)$ is monotone increasing from ϕ_0 to ϕ_m, there exists a constant $C' > 0$ such that

$\chi_0(C') = \eta$ and $\chi_0(y) \geq \eta$ for all $y \geq C'$. Using (3.35), we obtain

$$\begin{aligned} y &= \int_{\phi_0}^{\chi_0(y)} \frac{d\psi}{\sqrt{2(V(\psi) - V(\phi_m))}} , \\ &= \left\{ \int_{\phi_0}^{\eta} + \int_{\eta}^{\chi_0(y)} \right\} \frac{d\psi}{\sqrt{2(V(\psi) - V(\phi_m))}} , \\ &\leq C' + \int_{\eta}^{\chi_0(y)} \frac{d\psi}{\sqrt{k-\alpha}(\phi_m - \psi)} , \\ &\leq C' + \frac{1}{\sqrt{k-\alpha}} \left(\mathrm{Log}(\phi_m - \eta) - \mathrm{Log}(\phi_m - \chi_0(y)) \right) . \end{aligned}$$

This gives

$$\phi_m - \chi_0(y) \leq C_2 \exp(-C_1 y) .$$

■

Lemma 3.12 can be proved exactly in the same way.

Proof of Theorem 3.13 . In order to estimate $|\chi_\varepsilon - \chi_0|(y)$ for bounded y, we will consider χ_ε and χ_0 as solutions of two initial value problems for the nonlinear 2nd order differential equation

$$\frac{d^2\chi}{dy^2} = V'(\chi).$$

From (3.27) and $\dfrac{d\phi_\varepsilon}{dx}(0) = \delta_\varepsilon$, we deduce

$$\frac{d\chi_\varepsilon}{dy}(0) = \varepsilon\delta_\varepsilon .$$

On the other hand, (3.34) gives

$$\frac{d\chi_0}{dy}(0) = \alpha_0 = \sqrt{-2V(\phi_m)} = \varepsilon\delta_1 ,$$

which is a quantity that does not depend on ε.
Hence χ_ε and χ_0 can be considered respectively as the solutions of the initial value problems

$$\begin{cases} \dfrac{d^2\chi_\varepsilon}{dy^2} = V'(\chi_\varepsilon) , \quad y > 0 , \\ \chi_\varepsilon(0) = \phi_0 , \qquad \dfrac{d\chi_\varepsilon}{dy}(0) = \varepsilon\delta_\varepsilon \end{cases}$$

and

$$\begin{cases} \dfrac{d^2\chi_0}{dy^2} = V'(\chi_0) , \quad y > 0 , \\ \chi_0(0) = \phi_0 , \qquad \dfrac{d\chi_0}{dy}(0) = \varepsilon\delta_1 . \end{cases}$$

Using classical results on differential equations, we obtain that, for all $M > 0$, there exists a constant $C(M) > 0$ such that

$$|\chi_\varepsilon(y) - \chi_0(y)| \le C(M)(\varepsilon\delta_\varepsilon - \varepsilon\delta_1), \;\; 0 \le y \le M \tag{3.36}$$

In order to derive a uniform estimate on $[0, \frac{x_\varepsilon}{\varepsilon}]$, we next consider χ_ε and χ_0 as the respective solutions of the boundary value problems (3.28) and

$$\begin{cases} \dfrac{d^2\chi_0}{dy^2} \;=\; V'(\chi_0)\ , \quad 0 < y < \dfrac{x_\varepsilon}{\varepsilon}\ , \\ \chi_0(0) = \phi_0\ , \qquad \chi_0(\dfrac{x_\varepsilon}{\varepsilon}) \qquad \text{given}\ . \end{cases}$$

Hypotheses (3.8) and (3.9) or (3.26) imply that we can fix a $\gamma \le \phi_m$, $\phi_m - \gamma$ small enough, such that

$$V''(\psi) \ge 0 \quad \text{for all} \quad \psi \in [\gamma,\ \phi_m]\ . \tag{3.37}$$

We set

$$\tilde{\gamma} := \frac{\phi_m + \gamma}{2}\ .$$

For M large enough we have

$$y \ge M \Longrightarrow \tilde{\gamma} \le \chi_0(y) \le \phi_m\ .$$

By using (3.36) at $y = M$ and Lemma 3.8, we obtain for ε small enough

$$|\chi_\varepsilon(M) - \chi_0(M)| \le \frac{\phi_m - \gamma}{2}\ ,$$

which yields

$$\gamma \le \chi_\varepsilon(M)\ .$$

As χ_ε is increasing, we have

$$\gamma \le \chi_\varepsilon(y) \le \phi_m \quad \forall y \in [M,\ \frac{x_\varepsilon}{\varepsilon}]\ .$$

On the other hand we have also

$$\gamma \le \chi_0(y) \le \phi_m \quad \forall y \in [M,\ \frac{x_\varepsilon}{\varepsilon}]$$

and by (3.37) we have that $V''(\chi_\varepsilon(y)) \ge 0$ and $V''(\chi_0(y)) \ge 0$ for all $y \in [M,\ \frac{x_\varepsilon}{\varepsilon}]$.

We define the function

$$a_\varepsilon(y) = \begin{cases} \dfrac{V'(\chi_\varepsilon(y)) - V'(\chi_0(y))}{\chi_\varepsilon(y) - \chi_0(y)} & \text{if} \;\; \chi_\varepsilon(y) \ne \chi_0(y) \\ V''(\chi_0(y)) & \text{if} \;\; \chi_\varepsilon(y) = \chi_0(y) \end{cases}$$

The function $\chi_\varepsilon - \chi_0$ is solution of the elliptic problem

$$
\text{(3.38)} \qquad \begin{cases} -\dfrac{d^2}{dy^2}(\chi_\varepsilon - \chi_0) + a_\varepsilon(\chi_\varepsilon - \chi_0) &= 0 \ , \\ (\chi_\varepsilon - \chi_0)(M) &= \chi_\varepsilon(M) - \chi_0(M) \\ (\chi_\varepsilon - \chi_0)(\dfrac{x_\varepsilon}{\varepsilon}) &= \chi_\varepsilon(\dfrac{x_\varepsilon}{\varepsilon}) - \chi_0(\dfrac{x_\varepsilon}{\varepsilon}) \ . \end{cases}
$$

The function a_ε is continuous and $a_\varepsilon \geq 0$. By applying the maximum principle to the problem (3.38) we obtain

$$
|\chi_\varepsilon(y) - \chi_0(y)| \leq \ \text{Max}\ (|\chi_\varepsilon(\frac{x_\varepsilon}{\varepsilon}) - \chi_0(\frac{x_\varepsilon}{\varepsilon})|,\ |\chi_\varepsilon(M) - \chi_0(M)|)\ , \quad M \leq y \leq \frac{x_\varepsilon}{\varepsilon}
$$

and therefore, since $\chi_\varepsilon(\frac{x_\varepsilon}{\varepsilon}) = \phi_m$ and with (3.36)

$$
|\chi_\varepsilon(y) - \chi_0(y)| \leq \ \text{Max}\ (|\phi_m - \chi_0(\frac{x_\varepsilon}{\varepsilon})|,\ C(M)\varepsilon(\delta_\varepsilon - \delta_1))\ , \quad 0 \leq y \leq \frac{1}{\varepsilon}
$$

The estimate (3.32) can be deduced from Lemmas 3.11 and 3.8. The estimate (3.33) follows from Lemmas 3.12 and 3.9.

■

The analysis of the boundary layer in $x = 1$ is exactly the same.

Example: The Euler-Poisson Problem (continued) For the Euler-Poisson model we have that, for $u_0 > c_S$ and $u_0 < c_S$, all the boundary layers are "exponential", while, when $u_0 = c_S$, the boundary layer in $x = 1$ is "parabolic".

4 Non-monotone Solutions

We pass now to the existence of non-monotone solutions of the nonlinear Poisson problem (0.1). As in the monotone case, we first reduce the problem to finding $\delta = \frac{d\phi}{dx}(0)$ solution of an appropriate equation.

First remark that, if a solution of the nonlinear Poisson problem (0.1) is monotone-increasing on an interval $[x_1,\ x_2]$ from a value $\underline{\varphi}$ to $\overline{\varphi}$, Eq.(2.1) yields

$$
\frac{d\phi}{dx}(x) = +\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\phi(x))}\ , \quad \text{for all} \quad x \in [x_1,\ x_2]
$$

and therefore

$$
\text{(4.1)} \qquad x_2 - x_1 = \int_{\underline{\varphi}}^{\overline{\varphi}} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}}\ .
$$

In the same way, if in an interval $[x_1,\ x_2]$ the solution of (0.1) is monotone-decreasing from $\overline{\varphi}$ to $\underline{\varphi}$, we have

$$
\text{(4.2)} \qquad x_2 - x_1 = -\int_{\overline{\varphi}}^{\underline{\varphi}} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}}\ .
$$

Let $\hat{x}$ be the point where the first derivative of the solution vanishes. By (2.1) $\phi(\hat{x})$ and the parameter δ are linked by the relation

$$V(\phi(\hat{x})) = -\frac{\varepsilon^2\delta^2}{2} \tag{4.3}$$

and we must have

$$V(\phi(\hat{x})) = \operatorname*{Min}_{x\in[0,1]} V(\phi(x)) \ . \tag{4.4}$$

Let us now derive the equation satisfied by δ. We begin by considering the *non-monotone solutions of the* 1*st kind*, As in Lemma 2.1, we denote by $x_*(\delta)$ the stationary point where the solution reaches its minimum $\phi_*(\delta) < \phi_0$ (we shall more simply denote this point x_*^ε when the dependence in δ will not be considered). From Eq.(4.2) we have

$$x_*(\delta) = -\int_{\phi_0}^{\phi_*(\delta)} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} \ , \tag{4.5}$$

while Eq.(4.1) yields

$$1 - x_*(\delta) = \int_{\phi_*(\delta)}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} \ . \tag{4.6}$$

Hence a non-monotone solution of the first kind of the problem (0.1) exists if and only if one can find δ such that

$$-\int_{\phi_0}^{\phi_*(\delta)} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + \int_{\phi_*(\delta)}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} = 1 \ . \tag{4.7}$$

In the same way for the *solutions of the* 2*nd kind*, Eq.(4.1) yields

$$x^*(\delta) = \int_{\phi_0}^{\phi^*(\delta)} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}}$$

and from Eq.(4.2) we have

$$1 - x^*(\delta) = -\int_{\phi^*(\delta)}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}}$$

And a non-monotone solution of the second kind of the problem (0.1) exists if and only if one can find δ such that

$$\int_{\phi_0}^{\phi^*(\delta)} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} - \int_{\phi^*(\delta)}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} = 1 \ .$$

Analogously a *non-monotone solution of the* 3*rd kind* exists, if and only if one can find δ that solves the following equation.

$$-\int_{\phi_0}^{\phi_*(\delta)} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + \int_{\phi_*(\delta)}^{\phi^*(\delta)} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} - \int_{\phi^*(\delta)}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} = 1 \ . \tag{4.8}$$

4.1 Existence of Non-Monotone Solutions

In the following we restrict our study to the non-monotone solutions of the 1st kind. The cases of the solutions of 2nd and 3rd kind can be treated in the same way and, for them, we will only state the corresponding existence theorems at the end of this subsection. We have

Theorem 4.1. *The two following statements are equivalent:*

(i) *For ε small enough there exists at least one non-monotone solution of the first kind of Problem* (0.1).

(ii) *One of the two following properties is true:*

Case A. *There exists $\zeta \in [\phi_L,\ \phi_1] \cap \Phi_s$ such that $V(\zeta) = m(\phi_0,\ \phi_1)$ and there exists $\chi \in [\phi_L,\ \zeta)$ such that $V(\chi) < V(\zeta)$.*

Case B. *There exists $\zeta \in [\phi_L,\ \phi_1] \cap \Phi_s$ such that $V(\zeta) < m(\phi_0,\ \phi_1)$.*

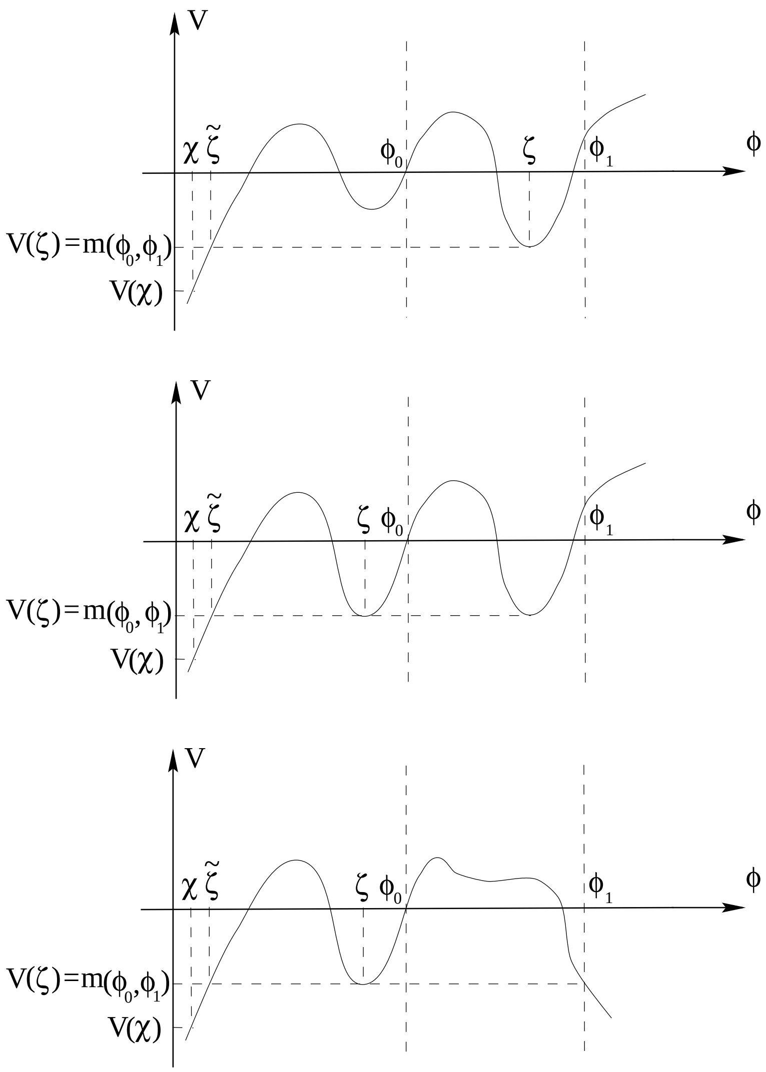


Fig. 4 : *Different situations corresponding to case A*

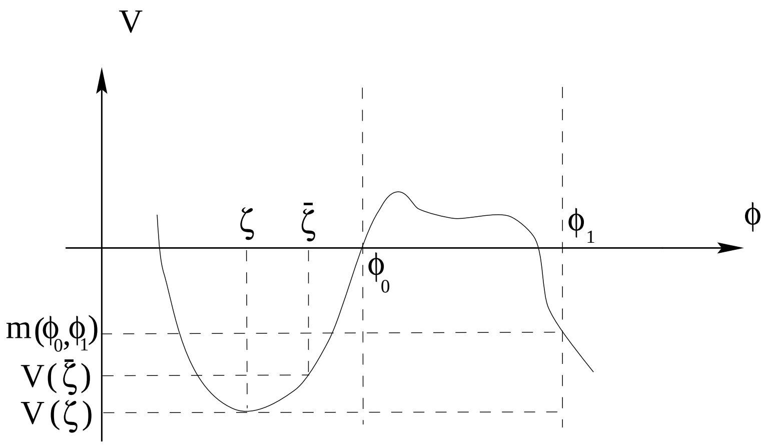


Fig. 5 : *Typical situation corresponding to case B*

Proof. First we prove $(ii) \Rightarrow (i)$.
We begin by considering the situation corresponding to case A and in which the hypotheses of case B are not fulfilled. We know that there exists $\zeta_0 \in [\phi_L, \phi_1] \cap \Phi_s$ such that $V(\zeta_0) = m(\phi_0, \phi_1)$ and there exists $\chi \in [\phi_L,\ \zeta_0)$ such that $V(\chi) < V(\zeta_0)$. Moreover, as

$$V(\chi) < V(\zeta_0) = \min_{[\phi_0, \phi_1]} V,$$

we have necessarily $\chi \in [\phi_L,\ \phi_0)$. Set

$$\zeta := \min\{\zeta' \in (\chi, \zeta_0] \cap \Phi_S, \quad \text{such that} \quad V(\zeta') = V(\zeta_0)\} \ . \tag{4.9}$$

Since the hypotheses of case B are not fulfilled, we know that

$$\text{there does not exist} \quad \zeta' \in [\chi, \zeta) \cap \Phi_S \quad \text{such that} \quad V(\zeta') \leq V(\zeta) \ . \tag{4.10}$$

We define $\zeta^{(1)} \in [\chi,\ \zeta]$ as the point such that $V(\zeta^{(1)}) = V(\zeta_0)$ and V is monotone on $(\chi, \zeta^{(1)}]$. Thanks to (4.10), such a point exists. Set

$$\delta_\chi := \sqrt{-\frac{2V(\chi)}{\varepsilon^2}} \quad \text{and} \quad \delta_\zeta := \sqrt{-\frac{2V(\zeta)}{\varepsilon^2}} \ .$$

Clearly

$$\delta_\zeta < \delta_\chi \ .$$

Moreover the monotonicity of V on $[\chi,\ \zeta^{(1)}]$ implies that, for all $\delta \in (\delta_\zeta,\ \delta_\chi)$ and $\varepsilon > 0$ fixed, Eq. (4.3) on the unknown ϕ,

$$V(\phi) = -\frac{\varepsilon^2 \delta^2}{2},$$

admits a unique solution $\phi_*(\delta) \in [\chi, \zeta^{(1)}]$ (recall that this value will be the minimum of ϕ on [0,1]). Hence Problem (4.7) is well-posed and can be written

$$2\int_{\phi_*(\delta)}^{\zeta^{(1)}} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + 2\int_{\zeta^{(1)}}^{\phi_0} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} = 1 \ .$$

A non-monotone solution of the first kind exists if we can solve this problem for a $\delta \in (\delta_\zeta,\ \delta_\chi)$. Let

$$f_1(\delta) := 2\int_{\zeta^{(1)}}^{\phi_0} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}}$$

and

$$g_1(\delta) := 2\int_{\phi_*(\delta)}^{\zeta^{(1)}} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}}\ .$$

Problem (4.7) writes

$$g_1(\delta) + f_1(\delta) = 1\ .$$

The function $f_1(\delta)$ is monotone decreasing and, since $V'(\zeta) = 0$, we have

$$f_1(\delta) \longrightarrow +\infty \quad \text{as} \quad \delta \to \delta_\zeta\ ,$$

while

$$f_1(\delta_\chi) = \frac{\varepsilon}{\sqrt{2}}\left[\int_{\zeta^{(1)}}^{\phi_0} \frac{d\psi}{\sqrt{V(\psi) - V(\chi)}} + \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{V(\psi) - V(\chi)}}\right]\ .$$

Since $V(\chi) < V(\zeta^{(1)}) = m(\phi_0, \phi_1)$, we know that there exists a constant $C > 0$ independent of ε such that

$$0 < f_1(\delta_\chi) < C\varepsilon\ .$$

We pass to the study of the function $g_1(\delta)$. We have

$$g_1(\delta_\zeta) \geq 0\ ,$$

while, for $\delta = \delta_\chi$,

$$g_1(\delta_\chi) = \frac{\varepsilon}{\sqrt{2}}\int_{\chi}^{\zeta^{(1)}} \frac{d\psi}{\sqrt{V(\psi) - V(\chi)}}\ .$$

By construction we can fix $\beta \in (\chi, \zeta^{(1)}]$ such that $\forall \phi \in [\chi, \beta]$, $V'(\phi) > 0$, then we set

$$\mu := \operatorname*{Min}_{\phi \in [\chi,\ \beta]} V'(\phi)\ .$$

We have

$$\begin{aligned} & V(\psi) - V(\chi) \geq \mu(\psi - \chi) \qquad && \forall \psi \in [\chi, \beta]\ , \\ \text{and}\quad & V(\psi) - V(\chi) \geq V(\beta) - V(\chi) && \forall \psi \in [\beta, \zeta^{(1)}]\ , \end{aligned}$$

which yield, naming C a generic positive constant

$$g_1(\delta_\chi) \leq \frac{\varepsilon}{\sqrt{2}}\int_{\chi}^{\beta} \frac{d\psi}{\sqrt{\mu}\sqrt{\psi - \chi}} + \frac{\varepsilon}{\sqrt{2}}\int_{\beta}^{\zeta^{(1)}} \frac{d\psi}{\sqrt{V(\beta) - V(\chi)}} \leq C\varepsilon\ . \tag{4.11}$$

Therefore, for ε small enough

$$g_1(\delta) + f_1(\delta) < \frac{1}{2} \quad \text{when} \quad \delta = \delta_\chi$$

and

$$g_1(\delta) + f_1(\delta) > \frac{3}{2} \quad \text{when} \quad \delta \to \delta_\zeta\ .$$

By a continuity argument, we can conclude that, for ε small enough, there exists a $\delta \in (\delta_\zeta,\ \delta_\chi)$ such that

$$g_1(\delta) + f_1(\delta) = 1 \ .$$

To complete the proof of $(ii) \Rightarrow (i)$ we still have to consider the situation corresponding to case B. By hypothesis, there exists a $\zeta \in [\phi_L, \phi_0) \cap \Phi_s$ such that $V(\zeta) < m(\phi_0, \phi_1)$. This implies that there exists $\bar{\zeta} \in (\zeta, \phi_0)$ such that

$$\begin{aligned}
&(1) \quad V'(\bar{\zeta}) > 0 \ , \\
&(2) \quad V(\bar{\zeta}) = m(\bar{\zeta}, \phi_1) \ , \\
&(3) \quad \forall \zeta' \in (\bar{\zeta}, \phi_1], \quad V(\zeta') > V(\bar{\zeta}) \ .
\end{aligned}$$

If necessary we change the point ζ redefining it as

$$\zeta := \mathrm{Max}\{\zeta' \in [\phi_L, \bar{\zeta}] \cap \Phi_s \text{ such that } \ V(\zeta') < V(\bar{\zeta})\} \ . \tag{4.12}$$

With this choice of ζ and $\bar{\zeta}$, $V(\phi)$ is monotone increasing in $(\zeta, \bar{\zeta})$. Set

$$\delta_{\bar{\zeta}} := \sqrt{-\frac{2V(\bar{\zeta})}{\varepsilon^2}} \ .$$

From (4.12) we have

$$\delta_{\bar{\zeta}} < \delta_\zeta \ .$$

By the monotonicity of V in $(\zeta, \bar{\zeta})$, for $\varepsilon > 0$ fixed and for any $\delta \in (\delta_{\bar{\zeta}},\ \delta_\zeta)$, Eq. (4.3) has a unique solution $\phi_*(\delta) \in (\zeta,\ \bar{\zeta})$. Hence we can rewrite (4.7) as

$$2\int_{\phi_*(\delta)}^{\bar{\zeta}} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + 2\int_{\bar{\zeta}}^{\phi_0} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} = 1 \ .$$

We set

$$f_2(\delta) := 2\int_{\bar{\zeta}}^{\phi_0} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}}$$

and

$$g_2(\delta) := 2\int_{\phi_*(\delta)}^{\bar{\zeta}} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} \ .$$

A non-monotone solution of the first kind exists if we can find a $\delta \in (\delta_{\bar{\zeta}},\ \delta_\zeta)$ such that

$$g_2(\delta) + f_2(\delta) = 1 \ . \tag{4.13}$$

It can be easily shown that there exist two positive constants independent of ε, both generically noted C, such that

$$0 < f_2(\delta) < C\varepsilon \quad \forall \delta \in (\delta_{\bar{\zeta}},\ \delta_\zeta) \tag{4.14}$$

and

$$0 < g_2(\delta_{\bar{\zeta}}) = C\varepsilon \ .$$

While, since $V'(\zeta) = 0$, we have

$$g_2(\delta) \longrightarrow +\infty \quad \text{as} \quad \delta \to \delta_\zeta \ .$$

As above, we can conclude that, for ε small enough, there exists a $\delta \in (\delta_{\bar{\zeta}},\ \delta_\zeta)$ that solves Eq. (4.13). Therefore a non-monotone solution of the first kind exists in this case.

We pass now to prove $(i) \Rightarrow (ii)$. The proof is by contradiction. Assume that the following statement is true:

for all $\zeta \in [\phi_L,\ \phi_1] \cap \Phi_s$, *(a)* $V(\zeta) \geq m(\phi_0,\ \phi_1)$
(b) If $V(\zeta) = m(\phi_0,\ \phi_1)$ then, for all $\chi < \zeta$, $V(\chi) \geq V(\zeta)$.

We want to show that this is in contradiction with the existence of a non-monotone solution of the first kind. Assume first that $[\phi_L,\ \phi_1] \cap \Phi_s = \emptyset$. If moreover

$$\text{for all} \quad \chi < \phi_0 \ , \quad V(\chi) > m(\phi_0,\ \phi_1) \ ,$$

then the problem of finding a δ that solves (4.7) is ill-posed as $\phi_*(\delta)$ cannot be defined: indeed it is impossible to find a $\phi_*(\delta) < \phi_0$ verifying Condition (4.4), *i.e.*, such that

$$V(\phi_*(\delta)) = \underset{x \in [0,\ 1]}{\text{Min}} \ V(\phi(x)) \ .$$

On the other hand, if

$$\text{there exists} \quad \chi < \phi_0 \quad \text{such that} \quad V(\chi) \leq m(\phi_0,\ \phi_1) \ ,$$

Problem (4.7) is now well-posed (since $\phi_*(\delta)$ is well defined), but it does not admit any solution for small ε. Actually, as the function $\delta^2 + \frac{2}{\varepsilon^2} V(\psi)$ can not vanish with zero derivative, we can find a constant $C > 0$ independent of ε such that

$$2 \int_{\phi_*(\delta)}^{\phi_0} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2} V(\psi)}} + \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2} V(\psi)}} \leq C\varepsilon \ .$$

Therefore, in this case, there does not exist any non-monotone solution of the first kind.

Now let us consider the case in which there exists $\zeta \in [\phi_L,\ \phi_1] \cap \Phi_s$ such that $V(\zeta) = m(\phi_0,\ \phi_1)$. By hypothesis

$$\text{for all} \quad \chi < \zeta \ , \quad V(\chi) \geq V(\zeta) \ .$$

As a matter of fact, as by assumption all the other possible minima are such that

$$V(\zeta) \geq m(\phi_0,\ \phi_1) \ ,$$

we have

$$V(\phi) \geq m(\phi_0,\ \phi_1) \quad \forall \phi \in [\phi_L,\ \phi_1] \ .$$

Therefore Condition (4.4) imposes

$$V(\phi_*(\delta)) = m(\phi_0,\ \phi_1) \ ,$$

which, using (4.3), fixes the value of δ as

$$\delta = \sqrt{-\frac{2m(\phi_0,\ \phi_1)}{\varepsilon^2}}\ .$$

Eq. (4.7) can not be solved by this δ as

$$2\int_{\phi_*(\delta)}^{\phi_0} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} = +\infty$$

and, again, we conclude that there does not exist any non-monotone solution of the first kind.

The last case to consider is the one in which

$$\text{for all} \quad \zeta \in [\phi_L,\ \phi_1] \cap \Phi_s\ , \quad V(\zeta) > m(\phi_0,\ \phi_1)\ .$$

If moreover

$$\text{for all} \quad \chi < \phi_0\ , \quad V(\chi) > m(\phi_0,\ \phi_1)\ ,$$

then Problem (4.7) is ill-posed as there does not exist any $\phi_*(\delta) < \phi_0$ verifying (4.4). While, if

$$\text{there exists} \quad \chi < \phi_0 \quad \text{for which} \quad V(\chi) \leq m(\phi_0,\ \phi_1)\ ,$$

we can define $\phi_*(\delta)$, but (4.7) can not be solved for small ε, as we can find a positive constant C independent of ε such that

$$2\int_{\phi_*(\delta)}^{\phi_0} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + \int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} < C\varepsilon\ .$$

Therefore there can not exist any non-monotone solution of the first kind and the assertion is proved. ■

In the proof of the above theorem we showed that, when hypotheses of case A hold, with a good choice of χ and ζ, we can define $\phi_*(\delta)$ and solve Problem (4.7) for $\delta \in (\delta_\zeta,\ \delta_\chi)$. In the same way, if hypotheses of case B hold, a good choice of ζ allows to define $\phi_*(\delta)$ and solve (4.7) for $\delta \in (\delta_{\bar{\zeta}},\ \delta_\zeta)$. Now it is natural to ask the question if other non-monotone solutions of the 1st kind can be found defining $\phi_*(\delta)$ and solving (4.7) for other ranges of δ. We state without proof the

Theorem 4.2. *Assume that a non-monotone solution of the first kind exists and that it corresponds to a $\delta \in (\delta_b,\ \delta_a)$ and to a $\phi_*(\delta) \in (\phi_a,\ \phi_b)$ ($\delta_{a,b} = \sqrt{-\frac{2V(\phi_{a,b})}{\varepsilon^2}}$). Then the two following statements are equivalent:*

(ii) *For ε small enough there exists at least one non-monotone solution of the first kind of Problem (0.1) corresponding to a $\delta > \delta_a$.*

(ii) *One of the two following properties is true:*

Case A. *There exists $\zeta' \in [\phi_L,\ \phi_1] \cap \Phi_s$ such that $V(\zeta') = m(\phi_a,\ \phi_1)$ and there exists $\chi \in [\phi_L,\ \zeta')$ such that $V(\chi) < V(\zeta')$.*

Case B. *There exists* $\zeta' \in [\phi_L,\ \phi_1] \cap \Phi_s$ *such that* $V(\zeta') < m(\phi_a,\ \phi_1)$.

This theorem can be proved in the same way of Theorem 4.1 when we let ϕ_a play the role of ϕ_0. It is now clear that we can iterate the reasoning we made in the proof of Theorem 4.1 and, given a function V, find different non-monotone solutions of the first kind.

Concerning the solutions of 2nd kind, we have

Theorem 4.3. *The two following statements are equivalent:*

(i) *For* ε *small enough there exists at least one non-monotone solution of the second kind of Problem* (0.1).

(ii) *One of the two following properties is true:*

Case A. *There exists* $\zeta \in [\phi_0,\ +\infty) \cap \Phi_s$ *such that* $V(\zeta) = m(\phi_0,\ \phi_1)$ *and there exists* $\chi \in (\zeta,\ +\infty)$ *such that* $V(\chi) < V(\zeta)$.

Case B. *There exists* $\zeta \in [\phi_0,\ +\infty) \cap \Phi_s$ *such that* $V(\zeta) < m(\phi_0,\ \phi_1)$.

And for the solutions of the 3rd kind, we can state

Theorem 4.4. *The two following statements are equivalent:*

(i) *For* ε *small enough there exists at least one non-monotone solution of the third kind of Problem* (0.1).

(ii) *One of the following properties is true:*

Case A. *There exists* $\zeta \in [\phi_L,\ +\infty) \cap \Phi_s$ *such that* $V(\zeta) = m(\phi_0,\ \phi_1)$ *and there exist* $\chi \in [\phi_L,\ \mathrm{Min}\{\zeta,\ \phi_0\})$, $\eta \in [\mathrm{Max}\{\phi_1,\ \zeta\},\ +\infty)$ *such that* $V(\chi) < V(\zeta)$, $V(\eta) < V(\zeta)$.

Case B.1 *There exists* $\zeta \in [\phi_L,\ \phi_0) \cap \Phi_s$ *such that* $V(\zeta) < m(\phi_0,\ \phi_1)$ *and there exists* $\eta \in [\phi_1,\ +\infty)$ *such that* $V(\eta) = V(\zeta)$.

Case B.2 *There exists* $\eta \in [\phi_1,\ +\infty) \cap \Phi_s$ *such that* $V(\eta) < m(\phi_0,\ \phi_1)$ *and there exists* $\zeta \in [\phi_L,\ \phi_0)$ *such that* $V(\zeta) = V(\eta)$.

The question of uniqueness of non-monotone solutions for fixed ε is open, but at the limit $\varepsilon \to 0$ it is possible, thanks to our constructive proof, to identify all the non-monotone limiting solutions in a precise way. We will present this result in the particular case of Euler-Poisson problem.

Example: The Euler-Poisson Problem (continued)

We begin by searching non-monotone solutions of the first kind. In the subsonic case the only stationary minimum of V is reached at ϕ_2. Therefore there can exist $\zeta \in [\phi_L,\ \phi_c] \cap \Phi_s$ such that $V(\zeta) \leq m(0,\ \phi_c)$ if and only if $\phi_c \geq \phi_2$ and $\zeta = \phi_2$. Under this assumption and when $\sqrt{\gamma\alpha} \leq u_0 \leq \lambda c_S$ (see Fig. 1.a), there does not exist any point $\chi < 0$ such that $V(\chi) < V(\phi_2)$, thus there does not exist any non-monotone solution of the 1st kind. On the other hand, when $\lambda c_S < u_0 < c_S$ and $\phi_c \geq \phi_2$, hypotheses of case A of Theorem 4.1 are fulfilled and a non-monotone solution of the 1st kind exists.

In the sonic and supersonic cases the only stationary minimum of V is at $\phi = 0$. Therefore there always exists $\zeta \in (\phi_L,\ \phi_c) \cap \Phi_s$ such that $V(\zeta) \leq m(0,\ \phi_c)$ and $\zeta = 0$. With a look at Fig. 1.e and 1.d it is easy to see that when $c_S \leq u_0 < \mu c_S$ there exists $\chi < 0$ such that $V(\chi) < V(\zeta)$, which implies that a non-monotone solution of the 1st kind exists in these cases. On the contrary, when $u_0 \geq \mu c_S$ (see Fig.. 1.c), $V(\psi) \geq V(0)\ \forall \psi < 0$ and there does not exist a non-monotone solution of the 1st kind.

Let us now consider the non-monotone solutions of the second kind. When $u_0 \geq c_S$, the function $V(\phi)$ is increasing for all $\phi > 0$. Therefore there can not exist any point $\chi \in (0,\ +\infty)$ such that $V(\chi) < V(0)$ and the hypotheses of Theorem 4.3 are never fulfilled.
When $u_0 < c_S$ and $\phi_c \geq \phi_2$, the same reasoning applies for the monotonicity of V for $\phi > \phi_2$. On the other hand, when $u_0 < c_S$ but $\phi_c < \phi_2$, we have $V(\phi_2) < m(0,\ \phi_c)$. Therefore hypotheses of case B of Theorem 4.3 are fulfilled and a non-monotone solution of the 2nd kind exists.

For the non-monotone solutions of the 3^{rd} kind it is now easy to verify that such a solution exists if and only if $\lambda c_S < u_0 < c_S$ and $\phi_c < \phi_2$. ■

5 Limit of Non-Monotone Solutions

We saw that the picture of the possible non-monotone solutions is quite complex: we had to consider three kinds of solutions and, for each one of them we did not have uniqueness in general. In the following we perform the asymptotic analysis in detail for the solutions of the first kind, the same analysis being possible for the other kinds of solutions.
In Section 3, where we performed the asymptotic analysis for the monotone solutions, we saw that a different role is played by the stationary points such that $V'' > 0$ and those for which $V'' = 0$. In the asymptotic analysis for the non-monotone solutions we implicitly admit that all the stationary points we consider are such that $V'' > 0$, the case $V'' = 0$ adding only some technical difficulties. Moreover, in Theorem 4.1, the analysis of the existence of non-monotone solutions of the 1st kind of Problem (0.1) consisted in the study of two main cases: Case A and Case B. In the study of the convergence of non-monotone solutions for $\varepsilon \to 0$ it is thus natural to analyze them separately.

5.1 Convergence Analysis. Case A

In this subsection -up to a change of (ζ, χ)-, we analyze the situation in which the hypotheses of Case A of Theorem 4.1 hold and (4.9) and (4.10) are verified. Moreover we use the same notations introduced in the proof of this Theorem. We introduce $\zeta^{(1)}$, denote $\delta_\varepsilon \in (\delta_\zeta,\ \delta_\chi)$ a solution of (4.7) for ε fixed small enough and set $\phi_*^\varepsilon := \phi_*(\delta_\varepsilon)$. Such a solution exists by Theorem 4.1.

Theorem 5.1. *Under the above assumption,*

$$\phi_*^\varepsilon \to \zeta^{(1)} \quad as \quad \varepsilon \to 0\ . \tag{5.1}$$

Proof. First we prove, by contradiction, that

$$V(\phi_*^\varepsilon) \to V(\zeta) \quad \text{as} \quad \varepsilon \to 0\ . \tag{5.2}$$

(4.3) yields

$$\delta_\varepsilon = \sqrt{-\frac{2V(\phi_*^\varepsilon)}{\varepsilon^2}} \ .$$

By construction $V(\phi_*^\varepsilon) < V(\zeta)$. Therefore the sequence $\{V(\phi_*^\varepsilon)\}$ is bounded, we can extract a converging subsequence. We assume that, at least for this subsequence

$$V(\phi_*^\varepsilon) \to K < V(\zeta) \ , \quad \text{as} \quad \varepsilon \to 0 \ . \tag{5.3}$$

For ε small enough

$$V(\zeta) - V(\phi_*^\varepsilon) > \frac{1}{2}\,|K - V(\zeta)| = C_1 > 0 \ .$$

Then, by (4.3),

$$\frac{\delta_\varepsilon^2 \varepsilon^2}{2} > C_1 - V(\zeta) \ .$$

This gives

$$\begin{aligned} f_1(\delta_\varepsilon) &= \frac{\varepsilon}{\sqrt{2}} \left\{ 2\int_{\zeta^{(1)}}^{\phi_0} + \int_{\phi_0}^{\phi_1} \right\} \frac{d\psi}{\sqrt{\dfrac{\delta_\varepsilon^2\varepsilon^2}{2} + V(\psi)}} \ , \\ &\leq \frac{\varepsilon}{\sqrt{2}} \left\{ 2\int_{\zeta^{(1)}}^{\phi_0} + \int_{\phi_0}^{\phi_1} \right\} \frac{d\psi}{\sqrt{C_1 + V(\psi) - V(\zeta)}} \ . \end{aligned}$$

By construction, we have $V(\zeta) = m(\zeta^{(1)},\ \phi_1)$, then

$$f_1(\delta_\varepsilon) \leq \frac{\varepsilon}{\sqrt{2}} \left\{ 2\int_{\zeta^{(1)}}^{\phi_0} + \int_{\phi_0}^{\phi_1} \right\} \frac{d\psi}{\sqrt{C_1}} \leq C\varepsilon \ , \quad C > 0 \quad \text{independent of } \varepsilon \ .$$

On the other hand we have

$$g_1(\delta_\varepsilon) = \varepsilon\sqrt{2} \int_{\phi_*^\varepsilon}^{\zeta^{(1)}} \frac{d\psi}{\sqrt{\dfrac{\delta_\varepsilon^2\varepsilon^2}{2} + V(\psi)}} \ .$$

As V is monotone in $(\chi,\ \zeta^{(1)})$, (5.3) yields

$$\phi_*^\varepsilon \to \phi_K \neq \zeta^{(1)} \ , \quad \text{for} \ \ \varepsilon \to 0$$

where $\phi_K \in (\chi,\ \zeta^{(1)})$ is the unique point such that $V(\phi_K) = K$. Moreover we have

$$V'(\phi_K) > 0 \ .$$

Therefore

$$\lim_{\varepsilon\to 0} \frac{g_1(\delta_\varepsilon)}{\varepsilon} = \sqrt{2} \int_{\phi_K}^{\zeta^{(1)}} \frac{d\psi}{\sqrt{V(\psi) - K}} < +\infty$$

and we have, for ε small enough

$$g_1(\delta_\varepsilon) \leq C\varepsilon \ , \quad C > 0 \quad \text{independent of } \varepsilon \ ,$$

which is absurd, for small ε, as

$$g_1(\delta_\varepsilon) + f_1(\delta_\varepsilon) = 1 \ .$$

This proves (5.2). To prove (5.1), it suffices to remember that by definition $\delta_\varepsilon \in [\chi,\ \zeta^{(1)}]$ and in this interval V is monotone. ■

Now, to continue the convergence analysis, as in Section 3, we would have to consider the general case in which the number of points $\zeta \in [\phi_L,\ \phi_1] \cap \Phi_s$ such that $V(\zeta) = m(\phi_0,\ \phi_1)$ is arbitrary. Once again we simplify the analysis assuming that this point is unique. The generalization to the case of multiple stationary points can be treated without additional difficulties.

Lemma 5.2. *Assume that Case A of Theorem* 4.1 *holds and that* ζ *verifies*

$$V''(\zeta) > 0 \text{ and } (4.10). \tag{5.4}$$

Then, as $\varepsilon \to 0$

$$x_*^\varepsilon \to \begin{cases} 1/2\ , & \text{if} \quad \zeta \notin [\phi_0,\ \phi_1]\ , \\ 1/3\ , & \text{if} \quad \zeta = \phi_0\ , \\ 0\ , & \text{if} \quad \zeta \in (\phi_0,\ \phi_1]\ , \end{cases}$$

where $x_*^\varepsilon = x_*(\delta_\varepsilon)$ *denotes the point where* ϕ^ε *takes its minimum.*

Proof. If we set

$$\alpha_\varepsilon = \sqrt{V(\zeta) - V(\phi_*^\varepsilon)} > 0$$

we can write (4.5) and (4.6) as

$$x_*^\varepsilon = \frac{\varepsilon}{\sqrt{2}} \int_{\phi_*^\varepsilon}^{\phi_0} \frac{d\psi}{\sqrt{\alpha_\epsilon^2 + (V(\psi) - V(\zeta))}}$$

and

$$1 - x_*^\varepsilon = \frac{\varepsilon}{\sqrt{2}} \int_{\phi_*^\varepsilon}^{\phi_1} \frac{d\psi}{\sqrt{\alpha_\epsilon^2 + (V(\psi) - V(\zeta))}}\ .$$

By the hypothesis (5.4), we have

$$k = V''(\zeta) > 0\ .$$

We want to apply Lemma 3.2 to each integral on a half-neighborhood of ζ, in particular, when $\zeta \in [\zeta^{(1)},\ \phi_0)$. We have

$$x_*^\varepsilon = \frac{\varepsilon}{\sqrt{2}} \left\{ \int_{\phi_*^\varepsilon}^{\zeta^{(1)}} + \int_{\zeta^{(1)}}^{\zeta} + \int_{\zeta}^{\phi_0} \right\} \frac{d\psi}{\sqrt{\alpha_\epsilon^2 + (V(\psi) - V(\zeta))}}\ ,$$

and (4.11) yields

$$\int_{\phi_*^\varepsilon}^{\zeta^{(1)}} \frac{d\psi}{\sqrt{\alpha_\epsilon^2 + (V(\psi) - V(\zeta))}} \leq C\ .$$

Thanks to (5.4), we can apply Theorem 5.1 and we have that $\alpha_\epsilon \to 0$ as $\varepsilon \to 0$. By Lemma 3.2 we get

$$x_*^\varepsilon \sim -\frac{\varepsilon}{\sqrt{2}} \frac{2}{\sqrt{k}} \operatorname{Log} \alpha_\epsilon$$

and, in the same way

$$1 - x_*^\varepsilon \sim -\frac{\varepsilon}{\sqrt{2}} \frac{2}{\sqrt{k}} \operatorname{Log} \alpha_\epsilon\ .$$

The factor 2 comes from the fact that $\phi_*^\varepsilon < \zeta < \phi_0$.
In the other cases we find

$$\begin{array}{llll} \text{if} \quad \zeta = \phi_0 \ , & x_*^\varepsilon \sim -\dfrac{\varepsilon}{\sqrt{2}}\dfrac{1}{\sqrt{k}}\operatorname{Log}\alpha_\epsilon & \text{and} & 1 - x_*^\varepsilon \sim -\dfrac{\varepsilon}{\sqrt{2}}\dfrac{2}{\sqrt{k}}\operatorname{Log}\alpha_\epsilon \ , \\ \text{if} \quad \zeta \in (\phi_0,\ \phi_1) \ , & x_*^\varepsilon \leq C\varepsilon & \text{and} & 1 - x_*^\varepsilon \sim -\dfrac{\varepsilon}{\sqrt{2}}\dfrac{2}{\sqrt{k}}\operatorname{Log}\alpha_\epsilon \ , \\ \text{if} \quad \zeta = \phi_1 \ , & x_*^\varepsilon \leq C\varepsilon & \text{and} & 1 - x_*^\varepsilon \sim -\dfrac{\varepsilon}{\sqrt{2}}\dfrac{1}{\sqrt{k}}\operatorname{Log}\alpha_\epsilon \ . \end{array}$$

The sequence $\{x_*^\varepsilon\}$ is bounded, then we can extract a subsequence converging to a $x_*(\delta_0)$, we have

$$\begin{array}{llll} \text{if} \quad \zeta \notin [\phi_0,\ \phi_1] \ , & x_*(\delta_0) = 1 - x_*(\delta_0) & \Rightarrow & x_*(\delta_0) = \dfrac{1}{2} \ , \\ \text{if} \quad \zeta = \phi_0 \ , & 2x_*(\delta_0) = 1 - x_*(\delta_0) & \Rightarrow & x_*(\delta_0) = \dfrac{1}{3} \ , \\ \text{if} \quad \zeta \in (\phi_0,\ \phi_1] \ , & & & x_*(\delta_0) = 0 \ . \end{array}$$

The limit of all subsequences being unique, the whole sequence converges to $x_*(\delta_0)$.

■

Following the same guidelines as in the convergence analysis for the monotone solutions, we give a result of uniform convergence of the sequence $\{\phi_\varepsilon\}$ and afterwards we pass to the boundary layers analysis.

Theorem 5.3. *Assume that hypothesis* (5.4) *holds. For all $a > 0$ arbitrarily small,*

$$\phi_\varepsilon \to \zeta \quad \textit{unif. in} \quad \begin{cases} [a,\ 1/2 - a] \cup [1/2 + a,\ 1 - a] & \textit{if} \quad \zeta \in [\phi_L,\ \phi_0) \ , \\ [a,\ 1/3 - a] \cup [1/3 + a,\ 1 - a] & \textit{if} \quad \zeta = \phi_0 \ , \\ [a,\ 1 - a] & \textit{if} \quad \zeta \in (\phi_0,\ \phi_1] \ , \\ [a,\ 1] & \textit{if} \quad \zeta = \phi_1 \ . \end{cases} \tag{5.5}$$

Proof. Consider the case $\zeta \in [\phi_L,\ \phi_0)$. We saw that the non-monotone solution of the first kind ϕ is monotone decreasing from $x = 0$ to x_* and monotone increasing after this value. From Lemma 5.2 we know that, for all $a > 0$, there exists a $\bar{\varepsilon}$ such that $\forall \varepsilon < \bar{\varepsilon}$, $x_*^\varepsilon \in (1/2 - a,\ 1/2 + a)$. Therefore, for ε small enough, $\phi(x)$ is monotone decreasing in $[0,\ 1/2 - a]$ and monotone increasing in $[1/2 + a,\ 1]$. Let us show that, for each $x \in (0,\ 1/2)$,

$$\phi_\varepsilon(x) \to \zeta \ .$$

As the sequence $\phi_\varepsilon(x)$ is bounded, we can extract a converging subsequence. Assume that this subsequence does not converge to ζ for an $x_1 \in (0,\ 1/2)$:

$$\phi_\varepsilon(x_1) \to \eta \neq \zeta \ , \qquad \text{as} \qquad \varepsilon \to 0 \ .$$

If $\eta > \zeta$, by (5.4) we have that, for all $\psi \in [\eta, \phi_0)$, $V(\psi) > V(\zeta)$. Therefore

$$x_1 = \frac{\varepsilon}{\sqrt{2}} \int_{\phi_\varepsilon(x_1)}^{\phi_0} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_*^\varepsilon)}} \sim \frac{\varepsilon}{\sqrt{2}} \int_{\eta}^{\phi_0} \frac{d\psi}{\sqrt{V(\psi) - V(\zeta)}} \leq C\varepsilon$$

which is absurd.
On the other hand, if $\eta < \zeta$, we have

$$\begin{aligned} 1 - x_1 &= \frac{\varepsilon}{\sqrt{2}} \left\{ \int_{\phi_*^\varepsilon}^{\phi_1} - \int_{\phi_*^\varepsilon}^{\phi_\varepsilon(x_1)} \right\} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_*^\varepsilon)}} \ , \\ &= 1 - x_*^\varepsilon - \frac{\varepsilon}{\sqrt{2}} \int_{\phi_*^\varepsilon}^{\phi_\varepsilon(x_1)} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_*^\varepsilon)}} \ . \end{aligned}$$

Lemma 5.2 yields

$$1 - x_*^\varepsilon \to 1/2 \ .$$

Moreover, by the hypothesis (5.4), for all $\psi \in (\phi_*^\varepsilon, \zeta)$, $V(\psi) > V(\phi_*^\varepsilon)$ and $V'(\phi_*^\varepsilon) > 0$. Since $\phi_\varepsilon(x_1) < \zeta$, for ε small enough, we have the estimate

$$\frac{\varepsilon}{\sqrt{2}} \int_{\phi_*^\varepsilon}^{\phi_\varepsilon(x_1)} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_*^\varepsilon)}} < C\varepsilon \ .$$

Thus

$$x_1 \geq \frac{1}{2} - C\varepsilon$$

which is absurd too. Therefore, necessarily the whole sequence verifies

$$\phi_\varepsilon(x) \to \zeta$$

for all $x \in (0,\ 1/2)$, which yields, by monotonicity,

$$\phi_\varepsilon \to \zeta \quad \text{unif. in} \quad [a,\ 1/2 - a] \ ,$$

by monotonicity.
In the same way, we can show that

$$\phi_\varepsilon \to \zeta \quad \text{unif. in} \quad [1/2 + a,\ 1 - a] \ .$$

Mimicking this proof, we find the results (5.5) in the other two cases $\zeta \in (\phi_0,\ \phi_1]$ and $\zeta = \phi_0$. ■

Therefore, outside boundary and free layers, ϕ_ε converges to the stationary minimum ζ. To complete the study of the limit $\varepsilon \to 0$, we perform now the analysis of these layers.

5.2 Asymptotic Analysis. Case A

Lemma 5.4. *Assume hypothesis* (5.4)*. Then we have for some constant $C > 0$ independent of ε*

$$0 < V(\zeta) - V(\phi_*^\varepsilon) \leq \exp(-\frac{C}{\varepsilon}) \ .$$

Proof. From the proof of Lemma 5.2 we have that, for ε small enough,

$$-\frac{\varepsilon}{\sqrt{2}}\frac{c}{\sqrt{k}}\,\mathrm{Log}\,\alpha_\varepsilon \geq \frac{1}{4}\ ,$$

which yields the desired result. ■

In a first time, we consider the case in which hypothesis (5.4) holds for a $\zeta \in [\phi_L,\ \phi_0)$. From Theorem 5.3 we know that the limit solution presents two boundary layers, in $x = 0$ and in $x = 1$, and a free layer in $x = 1/2$.

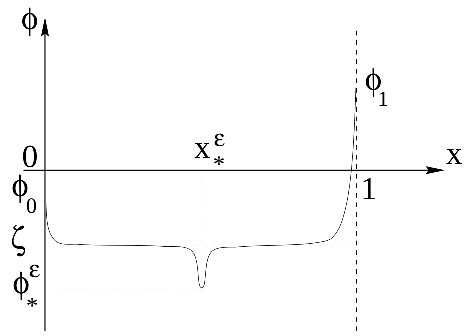


Fig. 6 : *Solution of the first kind for small ε. Case A.*

For which concerns the analysis of the boundary layers in $x = 0$ and in $x = 1$, the method of Section 3.4 applies and allows to conclude that both of these boundary layers are exponential. Therefore we limit our study to the free layer in $x_*(\delta_0)$. In particular, for the case we are studying $x_*(\delta_0) = x_*^0 = 1/2$. The analysis of this kind of layers is slightly different from the previous one, therefore we perform it in detail.
We know that, when ε is fixed and small and under the hypothesis (5.4), a non-monotone solution of the first kind is monotone decreasing from ϕ_0 to a value $\phi_*^\varepsilon \leq \zeta^{(1)} < \zeta$ and then monotone increasing from ϕ_*^ε to ϕ_1. Therefore there exist two points $0 < x_1^\varepsilon < x_2^\varepsilon < 1$ such that

$$\phi(x_1^\varepsilon) = \phi(x_2^\varepsilon) = \zeta\ .$$

With the same techniques of Lemma 5.2, it is possible to show that, in the case we are studying, we have

$$x_1^\varepsilon \to \frac{1}{4}\ , \quad x_2^\varepsilon \to \frac{3}{4}\ , \quad \text{when}\ \ \varepsilon \to 0\ .$$

We want to study the functions

$$\psi_\varepsilon^{(1)}(y) \ := \ \phi_\varepsilon(x_*^\varepsilon - \varepsilon y)\ , \qquad y = \frac{x_*^\varepsilon - x}{\varepsilon}\ , \qquad \text{if}\ \ x_1^\varepsilon \leq x \leq x_*^\varepsilon\ ,$$

$$\psi_\varepsilon^{(2)}(y) \ := \ \phi_\varepsilon(x_*^\varepsilon + \varepsilon y)\ , \qquad y = \frac{x - x_*^\varepsilon}{\varepsilon}\ , \qquad \text{if}\ \ x_*^\varepsilon \leq x \leq x_2^\varepsilon\ .$$

They are respectively solutions of the problems

$$(\mathrm{P}_\varepsilon^1)\left\{\begin{array}{l} \dfrac{d^2\psi_\varepsilon^{(1)}}{dy^2} \ = \ V'(\psi_\varepsilon^{(1)})\ , \qquad 0 < y < \dfrac{x_*^\varepsilon - x_1^\varepsilon}{\varepsilon}\ , \\[2ex] \dfrac{d\psi_\varepsilon^{(1)}}{dy}(0) = 0\ , \qquad \psi_\varepsilon^{(1)}\left(\dfrac{x_*^\varepsilon - x_1^\varepsilon}{\varepsilon}\right) = \zeta\ . \end{array}\right.$$

$$
(\mathrm{P}^2_\varepsilon)\begin{cases} \dfrac{d^2\psi_\varepsilon^{(2)}}{dy^2} = V'(\psi_\varepsilon^{(2)}) \ , & 0 < y < \dfrac{x_2^\varepsilon - x_\varepsilon^*}{\varepsilon} \ , \\ \dfrac{d\psi_\varepsilon^{(2)}}{dy}(0) = 0 \ , & \psi_\varepsilon^{(2)}\left(\dfrac{x_2^\varepsilon - x_\varepsilon^*}{\varepsilon}\right) = \zeta \ . \end{cases}
$$

The formal limit problems are, respectively

$$
(\mathrm{P}^1_0)\begin{cases} \dfrac{d^2\psi_0^{(1)}}{dy^2} = V'(\psi_0^{(1)}) \ , & y > 0 \ , \\ \dfrac{d\psi_0^{(1)}}{dy}(0) = 0 \ , & \lim\limits_{y\to+\infty} \psi_0^{(1)}(y) = \zeta \end{cases}
$$

and

$$
(\mathrm{P}^2_0)\begin{cases} \dfrac{d^2\psi_0^{(2)}}{dy^2} = V'(\psi_0^{(2)}) \ , & y > 0 \ , \\ \dfrac{d\psi_0^{(2)}}{dy}(0) = 0 \ , & \lim\limits_{y\to+\infty} \psi_0^{(2)}(y) = \zeta \ . \end{cases}
$$

We remark that Problems $(\mathrm{P}^1_{\varepsilon,0})$ and $(\mathrm{P}^2_{\varepsilon,0})$ are alike. Therefore, in the following, we perform the analysis only for the problems $(\mathrm{P}^1_{\varepsilon,0})$. This problem differs from the limit boundary layer problem (3.29) because of the Neumann boundary condition in $y = 0$. We begin by solving Problem (P^1_0)

Lemma 5.5. *There exists a monotone increasing solution of* (P^1_0).

Proof. Multiplying by $\dfrac{d\psi_0^{(1)}}{dy}$ the equation in (P^1_0) and integrating from 0 to y, we have

$$
\frac{d\psi_0^{(1)}}{dy}(y) = \sqrt{2(V(\psi_0^{(1)}(y)) - V(\psi_0^{(1)}(0)))}.
$$

Integrating again yields

$$
y = \int_{\psi_0^{(1)}(0)}^{\psi_0^{(1)}(y)} \frac{d\chi}{\sqrt{2(V(\chi) - V(\psi_0^{(1)}(0)))}} \ .
$$

Therefore $\psi_0^{(1)}(0)$ must be such that

$$
\lim_{\psi_0^{(1)}(y)\to\zeta} \int_{\psi_0^{(1)}(0)}^{\psi_0^{(1)}(y)} \frac{d\chi}{\sqrt{2(V(\chi) - V(\psi_0^{(1)}(0)))}} = +\infty \ ,
$$

Hence we must have $V(\psi_0^{(1)}(0)) = V(\zeta)$. Since we seek a monotone increasing function, we have $\psi_0^{(1)}(0) < \zeta$. This implies $\psi_0^{(1)}(0) = \zeta^{(1)}$ and the solution of (P^1_0) is implicitly defined by

$$
y = \int_{\zeta^{(1)}}^{\psi_0^{(1)}(y)} \frac{d\chi}{\sqrt{2(V(\chi) - V(\zeta))}} \ . \tag{5.6}
$$

■

Now we can show that the solution of the limit problem (P^1_0) $\psi^{(1)}_0$ converges exponentially to ζ when $y \to +\infty$.

Lemma 5.6. *The solution* $\psi^{(1)}_0$ *of problem* (P^1_0) *satisfies the inequality*

$$0 \le \zeta - \psi^{(1)}_0(y) \le C_1 \exp(-C_2\ y) \quad \textit{with} \quad C_1,\ C_2\ > 0 \quad \textit{independent of}\ \varepsilon\ .$$

Proof. We fix a $\kappa \in (0,\ V''(\zeta))$; there exists a $\tilde{\psi}$ such that, $\forall \psi \in [\tilde{\psi},\ \zeta]$,

$$V(\psi) - V(\zeta) \ge \frac{\kappa}{2}(\zeta - \psi)^2\ .$$

Let

$$C = \int_{\zeta^{(1)}}^{\tilde{\psi}} \frac{d\chi}{\sqrt{2(V(\chi) - V(\zeta))}}\ .$$

For $y > C$, from (5.6), we can write

$$\begin{aligned} y &= \left\{ \int_{\zeta^{(1)}}^{\tilde{\psi}} + \int_{\tilde{\psi}}^{\psi_0^{(1)}(y)} \right\} \frac{d\chi}{\sqrt{2(V(\chi) - V(\zeta))}}\ , \\ &\le C + \int_{\tilde{\psi}}^{\psi_0^{(1)}(y)} \frac{d\chi}{\sqrt{\kappa}(\zeta - \chi)}\ , \\ &\le C - C' \operatorname{Log}(\zeta - \psi^{(1)}_0(y))\ , \end{aligned}$$

which yields the desired result. ∎

Finally we have the estimate of the error between $\psi^{(1)}_\varepsilon(y)$ and $\psi^{(1)}_0(y)$.

Theorem 5.7. *There exists a constant $C > 0$ independent of ε such that, for all $y \in (0,\ \dfrac{x_1^\varepsilon}{\varepsilon})$, we have*

$$|\psi^{(1)}_0(y) - \psi^{(1)}_\varepsilon(y)| \le \exp(-\frac{C}{\varepsilon})\ .$$

Proof. The proof of this theorem is similar to Theorem 3.13's one, however some relevant differences encourage us to perform it in detail. In the sequel C denotes a generic positive constant independent of ε. From the definition of $\psi^{(1)}_\varepsilon$, we have

$$V(\psi^{(1)}_\varepsilon(0)) = V(\phi^\varepsilon_*) = -\frac{\varepsilon^2 \delta^2_\varepsilon}{2}\ .$$

On the other hand, Lemma 5.4 and the property $V'(\zeta^{(1)}) > 0$ yield

$$0 \le \zeta^{(1)} - \psi^{(1)}_\varepsilon(0) \quad \le \quad C(V(\zeta^{(1)}) - V(\psi^{(1)}_\varepsilon(0))) \quad \le \quad C \exp(-\frac{c}{\varepsilon})\ , \qquad C \ge 0\ .$$

We can consider $\psi^{(1)}_\varepsilon$ and $\psi^{(1)}_0$ respectively as solutions of the two initial values problems

$$\begin{cases} \dfrac{d^2\psi^{(1)}_\varepsilon}{dy^2} = V'(\psi^{(1)}_\varepsilon)\ , \qquad 0 < y < \dfrac{x^\varepsilon_* - x^\varepsilon_1}{\varepsilon}\ , \\[2ex] \psi^{(1)}_\varepsilon(0) = \phi^\varepsilon_*\ , \qquad \dfrac{d\psi^{(1)}_\varepsilon}{dy}(0) = 0\ . \end{cases}$$

$$\begin{cases} \dfrac{d^2\psi_0^{(1)}}{dy^2} = V'(\psi_0^{(1)}) \ , & y > 0 \ , \\ \psi_0^{(1)}(0) = \zeta^{(1)} \ , \qquad \dfrac{d\psi_0^{(1)}}{dy}(0) = 0 \ . & \end{cases}$$

By a standard result in differential equation theory we know that, for all $M > 0$, there exists a constant $C(M)$ such that, for ε small enough to have $\dfrac{x_*^\varepsilon - x_1^\varepsilon}{\varepsilon} > M$,

$$\begin{aligned} \forall y \in [0,\ M] \ , \qquad |\psi_\varepsilon^{(1)}(y) - \psi_0^{(1)}(y)| &\le C(M)|\zeta^{(1)} - \psi_\varepsilon^{(1)}(0)| \ , \\ &\le C'(M)\exp(-\frac{c}{\varepsilon}) \ . \end{aligned} \tag{5.7}$$

In order to derive a uniform estimate on $[0, \dfrac{x_*^\varepsilon - x_1^\varepsilon}{\varepsilon}]$, we next consider $\psi_\varepsilon^{(1)}$ and $\psi_0^{(1)}$ as the respective solutions of the Dirichlet boundary value problems

$$\frac{d^2\psi_\varepsilon^{(1)}}{dy^2} = V'(\psi_\varepsilon^{(1)}) \ , \qquad 0 < y < \frac{x_*^\varepsilon - x_1^\varepsilon}{\varepsilon}, \tag{5.8}$$

$$\psi_\varepsilon^{(1)}(0) = \phi_*^\varepsilon \ , \qquad \psi_\varepsilon^{(1)}(\frac{x_*^\varepsilon - x_1^\varepsilon}{\varepsilon}) = \zeta$$

and

$$\frac{d^2\psi_0^{(1)}}{dy^2} = V'(\psi_0^{(1)}) \ , \qquad 0 < y < \frac{x_*^\varepsilon - x_1^\varepsilon}{\varepsilon}, \tag{5.9}$$

$$\psi_0^{(1)}(0) = \zeta^{(1)} \ , \qquad \psi_0^{(1)}(\frac{x_*^\varepsilon - x_1^\varepsilon}{\varepsilon}) \text{ given.}$$

If we consider the difference between (5.8) and (5.9), we have

$$-\frac{d^2}{dy^2}(\psi_\varepsilon^{(1)} - \psi_0^{(1)})(y) + a_\varepsilon(\psi_\varepsilon^{(1)} - \psi_0^{(1)})(y) = 0 \ , \tag{5.10}$$

where

$$a_\varepsilon = \frac{V'(\psi_\varepsilon^{(1)}) - V'(\psi_0^{(1)})}{\psi_\varepsilon^{(1)} - \psi_0^{(1)}} \ .$$

Mimicking the proof of Theorem 3.13, we intend to use a maximum principle for the Equation (5.10), but, to do this, we must show that, at least for $y \ge M$, $a_\varepsilon \ge 0$. We note that there exists a $\beta \in (\zeta^{(1)},\ \zeta)$ such that, for $\beta \le \psi \le \zeta$, $V''(\psi) \ge 0$. By Lemma 5.6, we know that, choosing an M large enough, for $y \ge M$, $\psi_0^{(1)}(y) \ge \beta$. We need now to state an analogous result for $\psi_\varepsilon(y)$.

There exists a point y_β such that $\psi_\varepsilon(y_\beta) = \beta$ and, as $V'(\phi_*^\varepsilon) > \alpha > 0$, we have

$$y_\beta = \int_{\phi_*^\varepsilon}^{\beta} \frac{d\chi}{\sqrt{2(V(\chi) - V(\phi_*^\varepsilon))}} \le C \ ,$$

so it is bounded for all ε. As $\psi_\varepsilon(y)$ is monotone, we have

$$\psi_\varepsilon(y) \geq \beta \quad \text{for} \quad y \geq C\ . \tag{5.11}$$

Therefore, choosing $M \geq C$, we have

$$a_\varepsilon(y) \geq 0 \qquad \text{for} \qquad y \geq M$$

and we can use the maximum principle for the elliptic equation (5.10), which yields

$$|\psi_\varepsilon^{(1)} - \psi_0^{(1)}|(y) \leq \mathrm{Max}\left\{\left|\zeta - \psi_0^{(1)}\left(\frac{x_*(\delta_\varepsilon) - x_1^\varepsilon}{\varepsilon}\right)\right|,\ \left|\psi_\varepsilon^{(1)}(M) - \psi_0^{(1)}(M)\right|\right\}\ ,$$

and, from Lemma 5.6 and (5.7),

$$\begin{aligned}|\psi_\varepsilon^{(1)} - \psi_0^{(1)}|(y) &\leq \mathrm{Max}\left\{C_1 \exp(-\frac{C_2(x_*(\delta_\varepsilon) - x_1^\varepsilon)}{\varepsilon}),\ C'(M)\exp(-\frac{c}{\varepsilon})\right\} \\ &\leq \exp\left(-\frac{C}{\varepsilon}\right)\ .\end{aligned}$$

■

The same result being true for $\psi_\varepsilon^{(2)}$, We can conclude that the function ϕ_ε presents a double *exponential* free layer in x_*^0.
Using the same kind of analysis we can treat the cases in which $\zeta \in [\phi_0,\ \phi_1]$. Actually, if $\zeta = \phi_0$, the function ϕ_ε presents a double exponential free layer in $x_* \to 1/3$ and an exponential boundary layer in $x = 1$. If $\zeta \in (\phi_0,\ \phi_1]$, the solution presents a double exponential boundary layer in $x = 0$ and, with the exception of the case $\zeta = \phi_1$, an exponential boundary layer in $x = 1$.

5.3 Convergence Analysis. Case B

To complete the convergence analysis of the non-monotone solutions of the first kind we still have to consider the situation corresponding to case B of Theorem 4.1. We define ζ by (4.12) and use the same notations than in the proof of Theorem 4.1. For ε fixed and small we call $\delta_\varepsilon \in (\delta_{\bar\zeta},\ \delta_\zeta)$ a solution of (4.7) and $\phi_*^\varepsilon := \phi_*(\delta_\varepsilon)$. Such a solution exists for Theorem 4.1. We have

Theorem 5.8. *Assume the hypotheses of case B of Theorem* 4.1. *The sequence* ϕ_*^ε *is such that*

$$\phi_*^\varepsilon \to \zeta \text{ as } \varepsilon \to 0$$

Proof. By hypothesis $\delta_\varepsilon \in (\delta_{\bar\zeta},\ \delta_\zeta)$, so from (4.14) we deduce

$$f_2(\delta_\varepsilon) < C\varepsilon \quad C > 0 \quad \text{independent of } \varepsilon\ .$$

On the other hand

$$g_2(\delta_\varepsilon) = \varepsilon\sqrt{2}\int_{\phi_*^\varepsilon}^{\bar\zeta} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_*^\varepsilon)}}\ ,$$

with $\phi_*^\varepsilon \in [\zeta,\ \bar{\zeta}]$. We assume that

$$\phi_*^\varepsilon \to \phi_K > \zeta \tag{5.12}$$

and we show that this leads to a contradiction.
As V is monotone in the interval $(\zeta,\ \bar{\zeta})$, (5.12) gives

$$V(\phi_*^\varepsilon) \to K > V(\zeta)$$

and

$$V'(\phi_*^\varepsilon) > 0 \ .$$

Therefore the integral above converges and we have

$$g_2(\delta_\varepsilon) < C_2\varepsilon \quad C_2 > 0 \quad \text{independent of } \varepsilon \ ,$$

which is absurd since

$$g_2(\delta_\varepsilon) + f_2(\delta_\varepsilon) = 1$$

and the assertion is proved. ■

We saw that in the previous asymptotic analysis a fundamental result for the development of the study is the technical lemma 3.2 which gives the asymptotic behaviour of $f_1(\delta_\varepsilon)$ when $\varepsilon \to 0$. In the case we are considering here we need an analogous result concerning $g_2(\delta_\varepsilon)$. With this aim we state another technical lemma. The proof is given in Appendix. We remark that

$$\phi_*^\varepsilon \to \zeta$$

and we can use $(\phi_*^\varepsilon - \zeta)$ as a small parameter.

Lemma 5.9. *Consider a function $W \in \mathcal{C}^3([\zeta,\ \phi_r])$ such that $W'(\phi) > 0\ \forall \phi \in (\zeta,\ \phi_r)$ and*

$$W'(\zeta) = 0 \ , \qquad k := W''(\zeta) > 0 \ .$$

Moreover we take a sequence $\phi_\varepsilon \in (\zeta,\ \phi_r)$ such that

$$\phi_\varepsilon \to \zeta \qquad \textit{for} \qquad \varepsilon \to 0 \ .$$

We have

$$I_\varepsilon := \int_{\phi_\varepsilon}^{\phi_r} \frac{d\psi}{\sqrt{2(W(\psi) - W(\phi_\varepsilon))}} \sim -\frac{1}{\sqrt{k}} \operatorname{Log}(\phi_\varepsilon - \zeta) \ , \quad \textit{as} \quad \varepsilon \to 0 \ .$$

Lemma 5.10. *Assume the hypotheses of case B of Theorem 4.1. Then as $\varepsilon \to 0$*

$$x_*(\delta_\varepsilon) \to 1/2 \ .$$

Proof. We remember that, from Eq. (2.1), we have

$$x_*(\delta_\varepsilon) = \frac{\varepsilon}{\sqrt{2}} \left\{ \int_{\phi_*^\varepsilon}^{\bar{\zeta}} + \int_{\bar{\zeta}}^{\phi_0} \right\} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_*^\varepsilon)}}$$

and

$$1 - x_*(\delta_\varepsilon) = \frac{\varepsilon}{\sqrt{2}} \left\{ \int_{\phi_*^\varepsilon}^{\bar\zeta} + \int_{\bar\zeta}^{\phi_0} + \int_{\phi_0}^{\phi_1} \right\} \frac{d\psi}{\sqrt{V(\psi) - V(\phi_*^\varepsilon)}} .$$

Setting

$$k = V''(\zeta)$$

and applying Lemma 5.9 to each integral on $(\phi_*^\varepsilon,\ \bar\zeta)$, we have

$$x_*(\delta_\varepsilon) \sim -\frac{\varepsilon}{\sqrt{2k}} \operatorname{Log}(V'(\phi_*^\varepsilon))$$

and

$$1 - x_*(\delta_\varepsilon) \sim -\frac{\varepsilon}{\sqrt{2k}} \operatorname{Log}(V'(\phi_*^\varepsilon)) .$$

We conclude as for Lemmas 3.4 or 5.2. ∎

Mimicking the proof of Theorem 5.3, we can give a result of uniform convergence of the sequence $\{\phi_\varepsilon\}$.

Theorem 5.11. *Under the hypotheses of case B of Theorem 4.1, the non-monotone solution of the first kind ϕ_ε is such that, for all $a > 0$ arbitrarily small,*

$$\phi_\varepsilon \to \zeta \qquad \textit{unif. in } [a,\ 1-a] .$$

We remark that now we obtain a result of uniform convergence on $[a,\ 1-a]$ and not only on $[a,\ 1/2 - a] \cup [1/2 + a,\ \ 1-a]$ as was the case in the previous paragraph, because, thanks to Theorem 5.8, we have $\phi_*^\varepsilon \to \zeta$. Thus, this theorem states that, outside two boundary layers, ϕ_ε converges to the stationary minimum ζ. The analysis of these layers is not sensibly different from the one we made for the monotone solution.

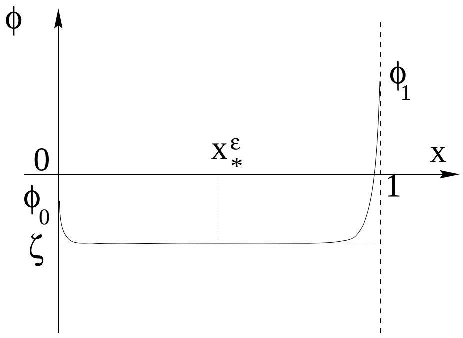


Fig. 7 : *Solution of the first kind for small ε. Case B*

Example: The Euler-Poisson Problem (continued)

In the previous discussion about non-monotone solutions of the Euler-Poisson Problem, we pointed out that a non-monotone solution of the first kind exists if and only if $c_S \le u_o < \mu c_S$ or $\lambda c_S < u_0 < c_S$ and $\phi_c \ge \phi_2$. In all these cases the $\zeta \in [\phi_L,\ \phi_c] \cap \Phi_s$ minimizing V verifies hypothesis $(ii.\mathrm{A})$ of Theorem 4.1. Therefore all the non-monotone solutions of the first kind

belong to the case A.
Looking at Figures 1.b and 1.d, it is easy to verify, by Theorems 5.1 and 5.3, that

$$\phi_*^\varepsilon \to \tilde{\zeta} = \phi_3 \quad \text{when} \quad \lambda c_S < u_0 < c_S \;\text{ and }\; \phi_c \geq \phi_2 \quad \text{or} \quad c_S < u_0 < \mu c_S$$

and, for all $a > 0$ arbitrarily small

$$\phi_\varepsilon \to \phi_2 \qquad \text{unif. on} \quad [a,\ 1-a] \qquad \text{if} \quad \lambda c_S < u_0 < c_S$$

$$\phi_\varepsilon \to 0 \quad \text{unif. on} \quad [a,\ \frac{1}{3} - a] \cup [\frac{1}{3} + a,\ 1-a] \quad \text{if} \quad c_S < u_0 < \mu c_S \ .$$

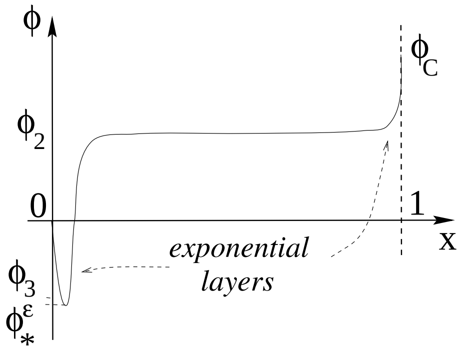


Fig. 8 : *Limit non-monotone solution of the first kind when* $\lambda c_S < u_0 < c_S$ *and* $\phi_c \geq \phi_2$

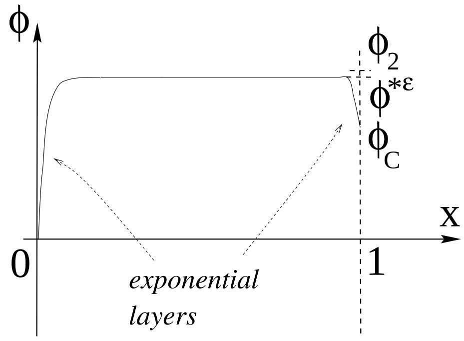


Fig. 9 : *Limit non-monotone solution of the first kind when* $c_S < u_0 < \mu c_S$

When $u_0 = c_S$, we have that $\zeta = 0$ is such that $V'(0) = V''(0) = 0$ and $V'''(0) > 0$. An analysis analogous to the one we made allows to conclude that

$$\phi_*^\varepsilon \to 0 \quad \text{when} \quad u_0 = c_S \ .$$

Therefore, in the sonic case, the non-monotone solutions of the first kind degenerate into a monotone solution when $\varepsilon \to 0$.

We saw that a non-monotone solution of the second kind exists if and only if

$$u_0 < c_S \quad \text{and} \quad \phi_c < \phi_2 \ .$$

In an analysis of the limit of non-monotone solutions of the second kind analogous to the one we made for the first kind ones, this case would correspond to the case B, as it verifies the hypothesis (ii.B) of Theorem 4.3. Therefore we have

$$\phi^{*\varepsilon} \to \phi_2 \quad \text{for} \quad \varepsilon \to 0$$

and, for all $a > 0$ arbitrarily small

$$\phi_\varepsilon \to \phi_2 \quad \text{unif. on} \quad [a,\ 1-a] \quad \text{for} \quad \varepsilon \to 0 \ .$$

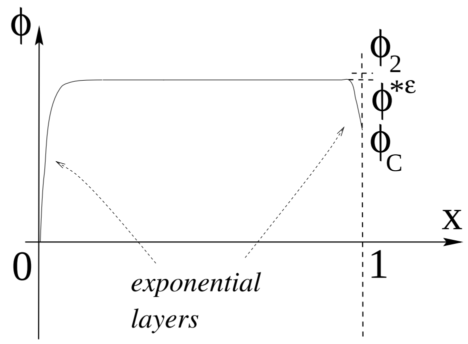


Fig. 10 : *Limit non-monotone solution of the second kind*

For which concerns the non-monotone solutions of the third kind, they exist if and only if $\lambda c_S < u_0 < c_S$ and $\phi_c < \phi_2$. We redraw the Sagdeev potential corresponding to this situation in Fig. 11. We have, with the notation of Section 2,

$$\phi_*^\varepsilon \to \phi_3 \qquad \text{for} \qquad \varepsilon \to 0$$

$$\phi^{*\varepsilon} \to \phi_2 \qquad \text{for} \qquad \varepsilon \to 0$$

and, for all $a > 0$ arbitrarily small

$$\phi_\varepsilon \to \phi_2 \quad \text{unif. on} \quad [a,\ 1-a] \quad \text{for} \quad \varepsilon \to 0\ .$$

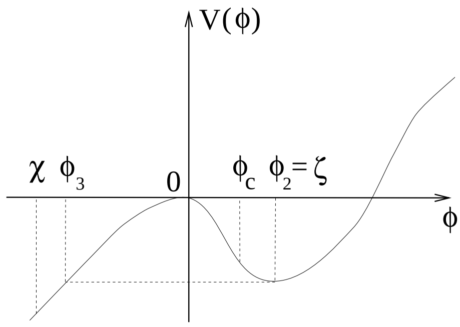


Fig. 11 : *Sagdeev potential when $\lambda c_S < u_0 < c_S$ and $\phi_c < \phi_2$*

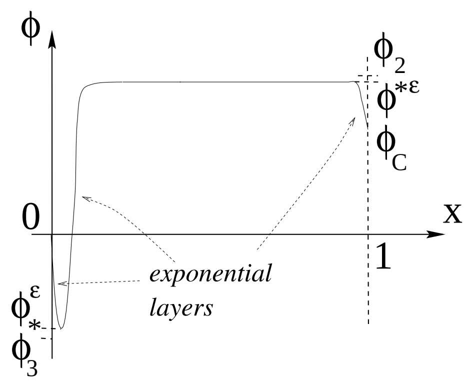


Fig. 12 : *Limit non-monotone solution of the third kind*

It is clear from the discussion above that all the limit non-monotone solutions we draw are determined in a unique way, as we announced at the end of Section 4.1.

∎

6 Periodic Solutions

We consider the problem of the existence, for ε fixed small, of periodic solutions of the nonlinear Poisson problem (0.1). From Lemma 2.1 we know that such a solution in one period T ($T < 1$) can present at most a local minimum ϕ_* and a local maximum ϕ^*.
Again we want to reduce the problem to the resolution of an equation in δ. To obtain this equation, as in Section 4, we integrate Eq. (2.1) separately on the increasing and decreasing branches of the solution $\phi(x)$.
First we remark that a periodic solution of NLP can verify the boundary conditions $\phi(0) = \phi_0$, $\phi(1) = \phi_1$, only if

$$\phi_* \leq \phi_0 \ , \qquad \phi^* \geq \phi_1 \ .$$

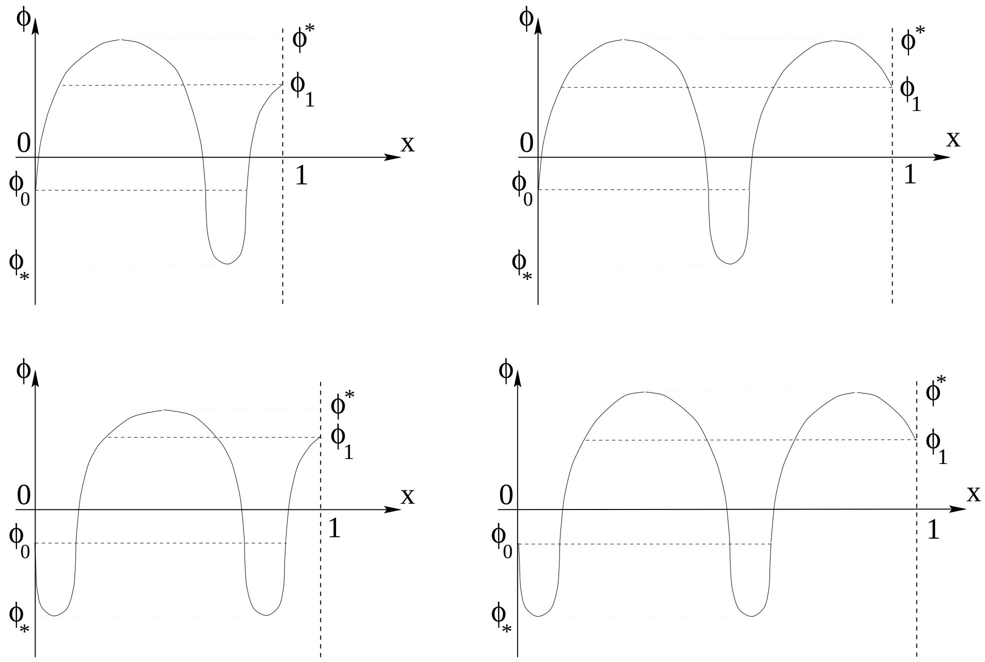


Fig. 13 : *Examples of Periodic Solutions*

Now, with the help of Fig. 13, it is easy to see that a generic periodic solution is composed of $2n+1$ ($n \in \mathbb{N}$) branches connecting ϕ_0 to ϕ_1 ($n+1$ increasing and n decreasing, with $2n$ equal to the integer part of $1/T$), of $p \in \mathbb{N}$ branches connecting $\phi_*(\delta)$ to ϕ_0 and of $q \in \mathbb{N}$ branches connecting ϕ_1 to $\phi^*(\delta)$. Moreover we have

$$p \;=\; \begin{cases} 2n\,, & \text{if } \dfrac{d\phi}{dx}(0) > 0\ . \\[2ex] 2(n+1)\,, & \text{if } \dfrac{d\phi}{dx}(0) < 0\ . \\[2ex] 0\,, & \text{if } \dfrac{d\phi}{dx}(0) = 0\ . \end{cases}$$

$$q \;=\; \begin{cases} 2n\,, & \text{if } \dfrac{d\phi}{dx}(1) > 0\ . \\[2ex] 2(n+1)\,, & \text{if } \dfrac{d\phi}{dx}(1) < 0\ . \\[2ex] 0\,, & \text{if } \dfrac{d\phi}{dx}(1) = 0\ . \end{cases}$$

The situation in which $\frac{d\phi}{dx}(0) = 0$ (resp. $\frac{d\phi}{dx}(1) = 0$) can correspond only to the choice $\phi_* = \phi_0$ (resp. $\phi^* = \phi_1$).

It is now easy to verify that a periodic solution of Problem (0.1) exists if and only if one can find δ such that

$$(6.1)\quad p\int_{\phi_*(\delta)}^{\phi_0} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + (2n+1)\int_{\phi_0}^{\phi_1} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} + q\int_{\phi_1}^{\phi^*(\delta)} \frac{d\psi}{\sqrt{\delta^2 + \frac{2}{\varepsilon^2}V(\psi)}} = 1\ .$$

We remark that if $\phi_* = \phi_0$ or $\phi^* = \phi_1$, the value of δ is fixed by the relation (4.3) to

$$\delta = \sqrt{-\frac{2}{\varepsilon^2}V(\phi_0)} \quad \text{or} \quad \delta = \sqrt{-\frac{2}{\varepsilon^2}V(\phi_1)}$$

and Eq. (6.1) can not be satisfied in general for a fixed ε.
Moreover it is worth noting the similarity between Eq. (6.1) and Eq. (4.8). Actually it is possible to prove that the conditions for the existence of a periodic solution of Problem (0.1) are exactly the same than the ones for the existence of a non-monotone solution of the 3rd kind. We have

Theorem 6.1. *The two following statements are equivalent:*

(i) *For ε small enough there exists a periodic solution of Problem* (0.1), *for all $n, p, q \in \mathbb{N}$.*

(ii) *One of the following properties is true:*

- **Case A.** *There exists $\zeta \in [\phi_L,\ +\infty) \cap \Phi_s$ such that $V(\zeta) = m(\phi_0,\ \phi_1)$ and there exist $\chi \in [\phi_L,\ \mathrm{Min}\{\zeta,\ \phi_0\})$, $\eta \in [\mathrm{Max}\{\phi_1,\ \zeta\},\ +\infty)$ such that $V(\chi) < V(\zeta)$, $V(\eta) < V(\zeta)$.*
- **Case B.1** *There exists $\zeta \in [\phi_L,\ \phi_0) \cap \Phi_s$ such that $V(\zeta) < m(\phi_0,\ \phi_1)$ and there exists $\eta \in [\phi_1,\ +\infty)$ such that $V(\eta) = V(\zeta)$.*

Case B.2 *There exists $\eta \in [\phi_1,\ +\infty) \cap \Phi_s$ such that $V(\eta) < m(\phi_0,\ \phi_1)$ and there exists $\zeta \in [\phi_L,\ \phi_0])$ such that $V(\zeta) = V(\eta)$.*

For what concerns the asymptotic analysis of this kind of solutions, once again one can obtain results similar to the ones of the previous section.

7 Appendix

Proof of Lemma 3.2 . Without loss of generality, from now on we assume $\phi > \phi_m$. We call ϕ_{m_r} other possible relative minima points of the function W in the interval $[\phi_m,\ \phi]$. For all $\beta \in (0,\ k)$, there exists a point $\chi = \chi_\beta$ such that

$$\frac{(k-\beta)}{2}(\psi - \phi_m)^2 \le W(\psi) \le \frac{(k+\beta)}{2}(\psi - \phi_m)^2 \qquad \text{for all} \qquad \psi \in (\phi_m,\ \chi)$$

and such that for all $\phi \in [\chi,\ \phi]$

$$W(\phi) \ge C_\chi = \min\{W(\chi),\ W(\phi_{m_r}),\ W(\phi)\} > 0\ . \tag{7.1}$$

Let us give upper and lower bounds for the integral

$$I_\alpha := \int_{\phi_m}^{\phi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} = \int_{\phi_m}^{\chi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} + \int_{\chi}^{\phi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}}\ .$$

On the one hand, we have

$$\begin{aligned}
I_\alpha &\ge \int_{\phi_m}^{\chi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} \\
&\ge \int_{\phi_m}^{\chi} \frac{d\psi}{\sqrt{\alpha^2 + (k+\beta)(\psi - \phi_m)^2}} \\
&\ge \frac{1}{\sqrt{k+\beta}} \operatorname{Log} \left| \frac{\sqrt{k+\beta}}{\alpha}(\chi - \phi_m) + \sqrt{1 + \frac{k+\beta}{\alpha^2}(\chi - \phi_m)^2} \right| \\
&\ge \frac{1}{\sqrt{k+\beta}} \operatorname{Log} \left(2\frac{\sqrt{k+\beta}}{\alpha}(\chi - \phi_m) \right) \\
&\ge \frac{1}{\sqrt{k+\beta}} \operatorname{Log} \left(2\sqrt{k+\beta}(\chi - \phi_m) \right) - \frac{1}{\sqrt{k+\beta}} \operatorname{Log}(\alpha)\ .
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
I_\alpha &\le \int_{\phi_m}^{\chi} \frac{d\psi}{\sqrt{\alpha^2 + (k-\beta)(\psi - \phi_m)^2}} + \int_{\chi}^{\phi} \frac{d\psi}{\sqrt{2C_\chi}} \\
&\le \frac{1}{\sqrt{k-\beta}} \operatorname{Log} \left| \frac{\sqrt{k-\beta}}{\alpha}(\chi - \phi_m) + \sqrt{1 + \frac{k-\beta}{\alpha^2}(\chi - \phi_m)^2} \right| + \frac{\phi - \chi}{\sqrt{2C_\chi}}\ .
\end{aligned}$$

By choosing $\alpha < \sqrt{k-\beta}(\chi - \phi_m)$,

$$\begin{aligned} I_\alpha &\leq \frac{\phi-\chi}{\sqrt{2C_\chi}} + \frac{1}{\sqrt{k-\beta}} \operatorname{Log}\left((1+\sqrt{2})\frac{\sqrt{k-\beta}}{\alpha}(\chi-\phi_m)\right) \\ &\leq \frac{\phi-\chi}{\sqrt{2C_\chi}} + \frac{1}{\sqrt{k-\beta}} \operatorname{Log}\left((1+\sqrt{2})\sqrt{k-\beta}(\chi-\phi_m)\right) - \frac{1}{\sqrt{k-\beta}} \operatorname{Log}\alpha \ . \end{aligned}$$

Therefore we have for $\beta \in (0,\ k)$ and α small enough

$$-\sqrt{\frac{k}{k+\beta}}\frac{\operatorname{Log}\left(2\sqrt{k+\beta}(\chi-\phi_m)\right)}{\operatorname{Log}\alpha} + \sqrt{\frac{k}{k+\beta}} \quad \leq \quad -\sqrt{k}\frac{I_\alpha}{\operatorname{Log}\alpha} \quad \leq$$

$$\leq \quad -\frac{\sqrt{k}}{\operatorname{Log}\alpha}\frac{\phi-\chi}{\sqrt{2C_\chi}} - \sqrt{\frac{k}{k-\beta}}\frac{\operatorname{Log}\left((1+\sqrt{2})\sqrt{k-\beta}(\chi-\phi_m)\right)}{\operatorname{Log}\alpha} + \sqrt{\frac{k}{k-\beta}} \ .$$

The lemma will be established if we show that, for all $\gamma > 0$ arbitrary small, we can find a number $\bar{\alpha} > 0$ such that, for all $0 < \alpha \leq \bar{\alpha}$

$$\left|\sqrt{k}\frac{I_\alpha}{\operatorname{Log}\alpha} + 1\right| < \gamma \ .$$

Actually, we can always choose $\beta \in (0,\ k)$ such that

$$\sqrt{\frac{k}{k+\beta}} - 1 > -\frac{\gamma}{2} \qquad \text{and} \qquad \sqrt{\frac{k}{k-\beta}} - 1 < \frac{\gamma}{2} \ .$$

We set

$$C_1(\beta) := \sqrt{\frac{k}{k+\beta}} \operatorname{Log}\left(2\sqrt{k+\beta}(\chi-\phi_m)\right)$$

and

$$C_2(\beta) := \sqrt{k}\frac{\phi-\chi}{\sqrt{2C_\chi}} + \sqrt{\frac{k}{k-\beta}} \operatorname{Log}\left((1+\sqrt{2})\sqrt{k-\beta}(\chi-\phi_m)\right) \ .$$

Clearly we can always find $\bar{\alpha} > 0$ such that, for all $\alpha \in (0,\ \bar{\alpha}]$, we have

$$\left|-\frac{C_1(\beta)}{\operatorname{Log}\bar{\alpha}} + \sqrt{\frac{k}{k+\beta}} - 1\right| < \gamma$$

and

$$\left|-\frac{C_2(\beta)}{\operatorname{Log}\bar{\alpha}} + \sqrt{\frac{k}{k-\beta}} - 1\right| < \gamma \ ,$$

which yields the desired result. ∎

Proof of Lemma 3.3 . For all $\beta \in (0,\ \theta)$, there exists a point $\chi = \chi_\beta \leq \phi$ such that

$$\frac{(\theta-\beta)}{6}(\psi-\phi_m)^3 \leq W(\psi) \leq \frac{(\theta+\beta)}{6}(\psi-\phi_m)^3 \qquad \text{for} \qquad \psi \in (\phi_m,\ \chi)$$

and such that for all $\phi \in [\chi,\ \phi]$

$$W(\phi) \geq \min\{W(\chi),\ W(\phi_{m_r}),\ W(\phi)\} > 0 \ ,$$

where, again, ϕ_{m_r} are the other possible minima points of the function W in the interval $[\phi_m,\ \phi_r]$. We derive an upper and a lower bound for the integral

$$I_\alpha := \int_{\phi_m}^{\phi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} = \int_{\phi_m}^{\chi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} + \int_{\chi}^{\phi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} \ .$$

First, we have

$$I_\alpha \geq \int_{\phi_m}^{\chi} \frac{d\psi}{\sqrt{\alpha^2 + 2W(\psi)}} \geq \int_{\phi_m}^{\chi} \frac{d\psi}{\sqrt{\alpha^2 + \frac{\theta+\beta}{3}(\psi - \phi_m)^3}} \ .$$

We set

$$\Lambda(u) = \int_0^u \frac{dt}{\sqrt{1+t^3}} \ .$$

This function is positive for positive u, monotone increasing and we have

$$\lim_{u \to \infty} \Lambda(u) = \Lambda_\infty < +\infty \ .$$

The above estimate can be written as

$$I_\alpha \geq \left[\frac{3}{\alpha(\theta+\beta)}\right]^{1/3} \Lambda\left[\left(\frac{\theta+\beta}{3\alpha^2}\right)^{1/3} (\chi - \phi_m)\right] \ . \tag{7.2}$$

On the other hand, we have

$$I_\alpha \leq \int_{\phi_m}^{\chi} \frac{d\psi}{\sqrt{\alpha^2 + \frac{\theta-\beta}{3}(\psi - \phi_m)^3}} + \int_{\chi}^{\phi} \frac{d\psi}{\sqrt{2W(\chi)}} \ ,$$

which yields

$$I_\alpha \leq \left[\frac{3}{\alpha(\theta-\beta)}\right]^{1/3} \Lambda\left[\left(\frac{\theta-\beta}{3\alpha^2}\right)^{1/3} (\chi - \phi_m)\right] + \int_{\chi}^{\phi} \frac{d\psi}{\sqrt{2W(\chi)}} \ . \tag{7.3}$$

By (7.2) and (7.3) we can easily show that for all $\gamma > 0$ arbitrary small, there exist $\beta > 0$, $\chi > 0$ and $\bar{\alpha} > 0$ such that

$$\forall \alpha \leq \bar{\alpha}, \qquad \sqrt[3]{3}\ \Lambda_\infty(1-\gamma) \leq (\alpha\theta)^{1/3} I_\alpha \leq \sqrt[3]{3}\ \Lambda_\infty(1+\gamma) \ ,$$

which yields the desired result. ∎

Proof of Lemma 5.9 . As $W \in \mathcal{C}^3$, $W''(\phi_\varepsilon) \to k$ as $\varepsilon \to 0$, that is, for all $\gamma > 0$, there exists a $\bar{\varepsilon}$ such that, $\forall \varepsilon \leq \bar{\varepsilon}$,

$$-\gamma \leq W''(\phi_\varepsilon) - k \leq \gamma \ ,$$

from now on we consider $\varepsilon \leq \bar{\varepsilon}$.
Moreover, for all $\beta \in (0,\ k/2)$, there exists a $\chi = \chi_\beta > 0$, such that, for all $\psi \in [\phi_\varepsilon,\ \phi_\varepsilon + \chi_\beta]$

$$
\begin{aligned}
W'(\phi_\varepsilon)(\psi - \phi_\varepsilon) + \frac{1}{2}(W''(\phi_\varepsilon) - \beta)(\psi - \phi_\varepsilon)^2 \leq \\
\leq W(\psi) - W(\phi_\varepsilon) \leq \\
\leq W'(\phi_\varepsilon)(\psi - \phi_\varepsilon) + \frac{1}{2}(W''(\phi_\varepsilon) + \beta)(\psi - \phi_\varepsilon)^2 \ .
\end{aligned}
$$

Choosing $\gamma \in (0, k/2)$, for ε small enough this yields

$$
\begin{aligned}
W'(\phi_\varepsilon)(\psi - \phi_\varepsilon) + \frac{1}{2}(k - \gamma - \beta)(\psi - \phi_\varepsilon)^2 \leq \\
\leq W(\psi) - W(\phi_\varepsilon) \leq \\
\leq W'(\phi_\varepsilon)(\psi - \phi_\varepsilon) + \frac{1}{2}(k + \gamma + \beta)(\psi - \phi_\varepsilon)^2 \ .
\end{aligned}
$$

We have

$$
I_\varepsilon \geq \int_{\phi_\varepsilon}^{\phi_\varepsilon + \chi} \frac{d\psi}{\sqrt{2W'(\phi_\varepsilon)(\psi - \phi_\varepsilon) + (k + \gamma + \beta)(\psi - \phi_\varepsilon)^2}} \ .
$$

Considering the change of variables

$$
y = \sqrt{(k + \gamma + \beta) + \frac{2W'(\phi_\varepsilon)}{\psi - \phi_\varepsilon}} \ ,
$$

the integral becomes

$$
\int_{\sqrt{(k+\gamma+\beta) + \frac{2W'(\phi_\varepsilon)}{\chi}}}^{+\infty} 2\frac{dy}{y^2 - (k + \gamma + \beta)} = \frac{1}{\sqrt{k + \gamma + \beta}} \operatorname{Log} \frac{\left(\sqrt{1 + \frac{2W'(\phi_\varepsilon)}{(k + \gamma + \beta)\chi}} + 1\right)^2}{\frac{2W'(\phi_\varepsilon)}{(k + \gamma + \beta)\chi}} \ .
$$

Therefore we have

$$
\begin{aligned}
I_\varepsilon \ & \geq \ \frac{1}{\sqrt{(k + \gamma + \beta)}} \operatorname{Log} \left\{ \frac{(k + \gamma + \beta)\chi}{2} \left(\sqrt{1 + \frac{2W'(\phi_\varepsilon)}{(k + \gamma + \beta)\chi}} + 1 \right)^2 \right\} - \frac{1}{\sqrt{(k + \gamma + \beta)}} \operatorname{Log}(W'(\phi_\varepsilon)) \ , \\
& \geq \ \frac{1}{\sqrt{(k + \gamma + \beta)}} \operatorname{Log} \{ 2(k + \gamma + \beta)\chi \} - \frac{1}{\sqrt{(k + \gamma + \beta)}} \operatorname{Log}(W'(\phi_\varepsilon)) \ .
\end{aligned}
$$

On the other hand, we have also

$$
\begin{aligned}
I_\varepsilon \ & \leq \ \int_{\phi_\varepsilon}^{\phi_\varepsilon + \chi} \frac{d\psi}{\sqrt{2(W(\psi) - W(\phi_\varepsilon))}} + \int_{\phi_\varepsilon + \chi}^{\phi_r} \frac{d\psi}{\sqrt{2(W(\psi) - W(\phi_\varepsilon))}} \ , \\
& \leq \ \int_{\phi_\varepsilon}^{\phi_\varepsilon + \chi} \frac{d\psi}{\sqrt{2W'(\phi_\varepsilon)(\psi - \phi_\varepsilon) + (k - \gamma - \beta)(\psi - \phi_\varepsilon)^2}} + C_\chi(\phi_r - \phi_\varepsilon - \chi) \ .
\end{aligned}
$$

Computing the integral exactly as above, we find

$$I_\varepsilon \;\leq\; C_\chi(\phi_r-\phi_\varepsilon-\chi)+\frac{1}{\sqrt{(k-\gamma-\beta)}}\,\mathrm{Log}\left\{\frac{(k-\gamma-\beta)\chi}{2}\left(\sqrt{1+\frac{2W'(\phi_\varepsilon)}{(k-\gamma-\beta)\chi}}+1\right)^2\right\}+$$

$$-\frac{1}{\sqrt{(k-\gamma-\beta)}}\,\mathrm{Log}(W'(\phi_\varepsilon))$$

and setting

$$M := \mathrm{Max}\;\, W'(\phi_\varepsilon)\;, \quad \text{for} \quad \varepsilon<\bar\varepsilon\;,$$

we have

$$I_\varepsilon \;\leq\; C_\chi(\phi_r-\phi_\varepsilon-\chi)+\frac{1}{\sqrt{(k-\gamma-\beta)}}\,\mathrm{Log}\left\{\frac{(k-\gamma-\beta)\chi}{2}\left(\sqrt{1+\frac{M}{(k-\gamma-\beta)\chi}}+1\right)^2\right\}+$$

$$-\frac{1}{\sqrt{(k-\gamma-\beta)}}\,\mathrm{Log}(W'(\phi_\varepsilon))\;.$$

We know that, for $\gamma\in(0,\;k/2)$, there exists $\varepsilon_1>0$ such that, for all $\varepsilon<\varepsilon_1$

$$(k-\gamma)(\phi_\varepsilon-\zeta)\leq W'(\phi_\varepsilon)\leq(k+\gamma)(\phi_\varepsilon-\zeta)\;.$$

Therefore we have, for $\beta\in(0,\;k/2)$ and ε small enough

$$-\sqrt{\frac{k}{k+\gamma+\beta}}\;\frac{\mathrm{Log}\left\{2\chi\dfrac{k+\gamma+\beta}{k+\gamma}\right\}}{\mathrm{Log}(\phi_\varepsilon-\zeta)}+\sqrt{\frac{k}{k+\gamma+\beta}}\quad\leq$$

$$\leq\quad -\sqrt{k}\frac{I_\varepsilon}{\mathrm{Log}(\phi_\varepsilon-\zeta)}\quad\leq$$

$$\leq\quad -\sqrt{\frac{k}{k-\gamma-\beta}}\;\frac{\mathrm{Log}\left\{\dfrac{(k-\gamma-\beta)\chi}{2(k-\gamma)}\left(\sqrt{1+\dfrac{M}{(k-\gamma-\beta)\chi}}+1\right)^2\right\}}{\mathrm{Log}(\phi_\varepsilon-\zeta)}+$$

$$-\frac{\sqrt{k}}{\mathrm{Log}(\phi_\varepsilon-\zeta)}C_\chi(\phi_r-\phi_\varepsilon-\chi)+\sqrt{\frac{k}{k-\gamma-\beta}}\;.$$

The Lemma will be established if we show that, for all $\eta>0$ arbitrarily small, we can find a number $\bar\varepsilon>0$ such that, for all $0<\varepsilon\leq\bar\varepsilon$

$$\left|-\sqrt{k}\frac{I_\varepsilon}{\mathrm{Log}(\phi_\varepsilon-\zeta)}-1\right|<\eta\;.$$

Actually, we can always choose a β and a γ in $(0,\;k/2)$ such that

$$\sqrt{\frac{k}{k+\gamma+\beta}}-1>-\frac{\eta}{2}\qquad\text{and}\qquad\sqrt{\frac{k}{k-\gamma-\beta}}-1<\frac{\eta}{2}\;.$$

We set

$$C_1(\beta) := \sqrt{\frac{k}{k+\gamma+\beta}}\,\mathrm{Log}\left\{2\chi\frac{k+\gamma+\beta}{k+\gamma}\right\}$$

and

$$C_2(\beta) := \sqrt{k}\,C_\chi(\phi_j-\phi_\varepsilon-\chi)+\sqrt{\frac{k}{k-\gamma-\beta}}\,\mathrm{Log}\left\{\frac{(k-\gamma-\beta)\chi}{2(k-\gamma)}\left(\sqrt{1+\frac{M}{(k-\gamma-\beta)\chi}}+1\right)^2\right\}\,.$$

It is clear that we can always find a $\bar{\varepsilon}$ such that, for all $\varepsilon \in (0,\ \bar{\varepsilon}]$,

$$\left|-\frac{C_1(\beta)}{\mathrm{Log}(\phi_\varepsilon-\zeta)}+\sqrt{\frac{k}{k+\gamma+\beta}}-1\right| < \eta$$

and

$$\left|-\frac{C_2(\beta)}{\mathrm{Log}(\phi_\varepsilon-\zeta)}+\sqrt{\frac{k}{k-\gamma-\beta}}-1\right| < \eta\ ,$$

which yields the desired result. ∎

Acknowledgements. This work has been supported by the GdR SPARCH of the CNRS. The first author acknowledges support from the TMR Individual Marie Curie Fellowship Programme.

References

[1] A. Ambroso, J. Gosset, F. Méhats, P.-A. Raviart, *On singular perturbation problems for the nonlinear Poisson equation*, course for the GdR SPARCH summer school in Oleron 15-20 September '97.

[2] N. Ben Abdallah, S. Mas-Gallic, P.-A. Raviart, *A Mathematical Analysis of Electric Probe Models*, Transp. Theo. Stat. Phys., vol. 25, pp. 263-281 (1996)

[3] Diener F., *Sauts des Solutions des Équations* $\varepsilon\ddot{x} = f(t,x,\dot{x})$, SIAM J. Math. Anal., vol. 17, No.3, pp. 533-559 (1986).

[4] O'Malley Jr. R.E., *Singular Perturbation Methods for Ordinary Differential Equations*, Springer-Verlag (1991).

[5] K.-U. Riemann, *The Bohm criterion and sheath formation*, Review article, J. Phys. D: Appl. Phys., vol. 24, pp. 493-518 (1991).