Adiabatic approximation of the Schrödinger-Poisson system with a partial confinement

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Abstract

Asymptotic quantum transport models of a two-dimensional electron gas are presented. The starting point is a singular perturbation of the three-dimensional Schrödinger-Poisson system. The small parameter ε is the scaled width of the electron gas and appears as the lengthscale on which a one dimensional confining potential varies. The rigorous $\varepsilon \to 0$ limit is performed by projecting the three dimensional wavefunction on the eigenfunctions corresponding to the confining potential. This leads to a two-dimensional Schrödinger-Poisson system with a modified Poisson equation keeping track of the third dimension. This limit model is proven to be a first-order approximation of the initial model. An intermediate model, called the "2.5D adiabatic model" is then introduced. It shares the same structure as the limit model but is shown to be a second-order approximation of the 3D model.

Key words : Adiabatic approximation, Energy estimates, Strichartz' estimates, error estimates, nonlinear analysis, two dimensional electron gas.

1 Introduction

Systems with reduced dimensionality are the basis of operation of most of nanoscale electronic devices. Among them, is the two-dimensional electron gas (2DEG) [1, 2, 9], in which the electrons are strongly confined in one direction so that collisionless transport is allowed in the two remaining ones. Although the transport is quasi bidimensional, the Coulomb interaction results in a fully three dimensional structure. Indeed, the particle density is a sheet density concentrated on the two-dimensional electron gas plane, which generates through mean field interaction a fully three dimensional potential. In [17], an approximate Schrödinger-Poisson model taking into account the quasi-bidimensional nature of electron transport while keeping a three dimensional description of the electrostatic potential, was proposed and numerically implemented in the stationary framework for electron waveguide structures. The model has been shown numerically to be in a very good agreement with the fully three dimensional Schrödinger-Poisson system, while having a much lower numerical complexity. The aim of this paper is to prove by a rigorous asymptotic analysis that the model introduced in [17] is a good approximation of the fully three-dimensional model and to quantify the discrepancy between the two models. In order to simplify the setting and to avoid additional technicalities induced by stationarity and by boundary effects, we shall consider the time-dependent problem in the whole space. The case of stationary boundary value problems will be the subject of a forthcoming work by the third author of this paper [16].

Denoting by z the confined direction, we shall consider the following singularly perturbed Schrödinger-Poisson system:

$$i\partial_t\psi^\varepsilon = -\frac{1}{2}\Delta_{x,z}\,\psi^\varepsilon + \frac{1}{\varepsilon^2}V_c\left(\frac{z}{\varepsilon}\right)\psi^\varepsilon + V^\varepsilon\psi^\varepsilon \tag{1.1}$$

$$\psi^\varepsilon(0,x,z) = \psi_0^\varepsilon(x,z) \tag{1.2}$$

$$V^\varepsilon = \frac{1}{4\pi r} * \left(|\psi^\varepsilon|^2\right), \tag{1.3}$$

where $x \in \mathbb{R}^2$, $z \in \mathbb{R}$, $r = \sqrt{|x|^2 + z^2}$, the potential V^ε is the selfconsistent potential due to space charge effects and the external confinement potential $V_c^\varepsilon(z) = \frac{1}{\varepsilon^2}V_c\left(\frac{z}{\varepsilon}\right)$ is given. In this work, the asymptotic behavior of the solution of this nonlinear system is studied when ε goes to 0. Two approximate models are exhibited: the limit model (*2D surface density model*) and an intermediate ε-dependent model (*2.5D adiabatic model*), which is shown to be a more accurate approximation of the initial model.

Quantum systems confined on a surface have been studied previously in [8, 10, 15, 21]. Starting from a similar scaling on the transverse Hamiltonian, these authors consider the linear Schrödinger equation with a confinement on a general surface and derive an effective Hamiltonian which locally depends on the curvature properties of the surface. In our case, the effective Hamiltonian at the leading order is trivial since the surface is the plane $z = 0$. The main difficulty here stems from the nonlinear character of the problem due to the selfconsistent potential.

As remarked in [21], quantum constrained systems can be linked to the Born-Oppenheimer approximation in molecular dynamics [12, 19, 21]. In order to analyze this link, let us rescale the variables z, t by setting $\tilde{z} = \frac{z}{\varepsilon}$, $\tilde{t} = \frac{t}{\varepsilon}$ and let $\tilde{x} = x$. To keep densities of order $\mathcal{O}(1)$, we also need to rescale ψ by a factor $\frac{1}{\sqrt{\varepsilon}}$, hence the selfconsistent potential is rescaled by $\frac{1}{\varepsilon}$. Denoting again (with an abuse of notation) by ψ^ε and V^ε the functions of the new variables, the system takes the form

$$i\varepsilon\partial_{\tilde{t}}\psi^\varepsilon = -\frac{\varepsilon^2}{2}\Delta_{\tilde{x}}\psi^\varepsilon - \frac{1}{2}\partial_{\tilde{z}}^2\psi^\varepsilon + (V_c + \varepsilon V^\varepsilon)\psi^\varepsilon. \tag{1.4}$$

The above problem (in the linear case) has been studied in particular in [3, 19]. However, the problem (1.1)–(1.3) is not just a rescaling of the Born-Oppenheimer asymptotics for two reasons. The first reason is, again, the nonlinear character of this system, which might induce rapid time oscillations of V^ε. The second reason is the time scale. Indeed, if the asymptotics is done for times $\tilde{t}$ of order 1 for the Born-Oppenheimer problem (1.4), then t is of order ε in the initial problem (1.1)–(1.3). Therefore, since we are here interested in time intervals of order 1 for the variable t, working in the variable $\tilde{t}$ would necessitate longer time intervals (of the order of $1/\varepsilon$) which is more difficult. The two problems share however similar properties of adiabatic decoupling. The systems can be diagonalized by using the eigenspaces of the transverse Hamiltonian $-\frac{1}{2}\partial_z^2 + V$ (in which t and x are frozen). Within each eigenspace the dynamics is governed by an effective potential and is quantum in our case, wheras semiclassical behaviour is expected in the Born-Oppenheimer approximation.

The paper is organized as follows. In Section 2, we first make precise the properties of the confinement operator, and define the two approximate models (namely the 2D and the 2.5D models). Then we state the main results of this paper, namely Theorems 2.5, 2.6 and 2.7. Section 3 is devoted to the proof of ε-independent estimates for (1.1)–(1.3). In Section 4, we put both approximate models into a more general framework allowing to prove existence and uniqueness of their solutions. The 2.5D adiabatic model is shown to be a second order approximation in Section 5, while in Section 6 the 2D surface density model is proven to be only a first order approximation. Finally, the appendix contains some basic results on the Schrödinger equation and the Poisson equation which are used all along the paper.

Remark on the scaling. Before going further, and in order to make clear the physical assumptions made here, let us show how the system (1.1)–(1.3) can be obtained by a rescaling of the Schrödinger-Poisson system written in the physical dimensional variables. Let $\Psi(T,X,Z)$, $\mathcal{V}(T,X,Z)$ be the solution of

$$i\hbar\partial_T\Psi = -\frac{\hbar^2}{2m}\Delta_{X,Z}\,\Psi + (\mathcal{V}_c + \mathcal{V})\Psi \tag{1.5}$$

$$\mathcal{V} = \frac{e^2}{4\pi\varepsilon_M}\,\frac{1}{\sqrt{|X|^2+Z^2}} * \left(|\Psi|^2\right), \tag{1.6}$$

where m is the effective mass, e is the elementary charge of the electrons and ε_M is the electric permittivity of the material. We introduce two characteristic energies, E_c and E, which are respectively the typical energy of the confinement and the typical kinetic energy of the electrons. The assumption of a strong confinement is

$$\varepsilon^2 = \frac{E}{E_c} \ll 1. \tag{1.7}$$

The confinement operator is the partial Hamiltonian defined on $\mathbb{R}$ by $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial Z^2} + \mathcal{V}_c$. Hence we deduce that the typical length L_c of the confinement, defined as the spatial extension of the eigenvalues of this operator, satisfies $\frac{\hbar^2}{2mL_c^2} = E_c$, and the confinement potential takes the form $\mathcal{V}_c(Z) = E_cV_c(\frac{Z}{L_c})$, where V_c denotes a dimensionless potential. Since we are interested in quantum models for the transport of the electrons, the typical space length L and the typical time $\mathcal{T}$ are deduced from the kinetic energy (this crucial assumption says that the initial data are not oscillating): $\frac{\hbar}{\mathcal{T}} = \frac{\hbar^2}{2mL^2} = E$, thus (1.7) gives $\frac{L_c}{L} = \varepsilon$. Finally, we assume that the selfconsistent potential is of the same order of magnitude as the kinetic energy, which means that if N_0 is the typical density (the scale of $|\Psi|^2$), we have

$$\frac{e^2 N_0 L^2}{\varepsilon_M} = E.$$

With these assumptions, setting

$$t = \frac{T}{\mathcal{T}}, \qquad (x,z) = \left(\frac{X}{L}, \frac{Z}{L}\right), \qquad \psi^\varepsilon = \frac{\Psi}{\sqrt{N_0}}, \qquad V^\varepsilon = \frac{\mathcal{V}}{E},$$

the system (1.5)-(1.6) is written (1.1)–(1.3) in the dimensionless variables.

2 Notations and main results

Throughout this paper, for any $q \in [1, \infty]$, we shall denote by q' its conjugate and for any $q \in [2, \infty]$, we denote by q^* its 2-conjugate, respectively defined by

$$q' = \frac{q}{q-1} \qquad ; \qquad q^* = \frac{2q}{q-2}.$$

We define the following functional spaces:

Definition 2.1 *Let $1 \le p, q, r \le +\infty$. The spaces $L^p_x L^q_z$ and $L^r_t L^p_x L^q_z$ are defined by*

$$L^p_x L^q_z(\mathbb{R}^3) = \left\{ u \in L^1_{loc}(\mathbb{R}^3), \quad \|u\|_{L^p_x L^q_z(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^2} \|u(x,\cdot)\|^p_{L^q(\mathbb{R})} \, dx \right)^{1/p} < +\infty \right\}$$

(with an obvious generalization of this definition for $p = +\infty$),

$$L^r_t L^p_x L^q_z((0,T) \times \mathbb{R}^3) = L^r((0,T), L^p_x L^q_z(\mathbb{R}^3)).$$

When there is no ambiguity, we shall simply denote these spaces by $L^p_x L^q_z$ and $L^r_t L^p_x L^q_z$ and the corresponding norms by $\|\cdot\|_{p,q}$ and by $\|\cdot\|_{r,p,q}$ (when there are two indices, the variables are (x,z); when there are three indices, the variables are (t,x,z)).

For a function $f = f(z)$ belonging to $L^1(\mathbb{R})$, we denote $\langle f \rangle = \int_{\mathbb{R}} f(z) dz$. In particular, if $n(t,x,z)$ is the particle density, the surface particle density is defined by $n_s(t,x) = \langle n(t,x,\cdot) \rangle = \int_{\mathbb{R}} n(t,x,z) \, dz$.

The symbol $*$ denotes a convolution with respect to all the variables $(x,z) \in \mathbb{R}^3$; partial convolutions are denoted by $*_x$ and $*_z$.

2.1 Properties of the confinement operator

Let us now introduce the basic assumptions made on the confining potential.

Assumption 2.2 *(i) The rescaled confining potential $V_c = V_c(z)$ is a nonnegative real-valued function in $L^2_{loc}(\mathbb{R})$.*
(ii) The operator $A = -\frac{1}{2}\frac{d^2}{dz^2} + V_c$ defined on $L^2(\mathbb{R})$ with the domain

$$\mathcal{D}(A) = \left\{u \in H^2(\mathbb{R}) \text{ such that } V_c\, u \in L^2(\mathbb{R})\right\}$$

admits a nondegenerate eigenvalue E associated to an eigenfunction $\chi(z)$ such that $z\chi \in L^2(\mathbb{R})$.

The first part of this assumption implies that the operator A is self-adjoint and nonnegative (see *e.g.* [18]). The partial Hamiltonian involved in (1.1) is obtained by rescaling the operator A:

$$A^\varepsilon = -\frac{1}{2}\frac{d^2}{dz^2} + V_c^\varepsilon = -\frac{1}{2}\frac{d^2}{dz^2} + \frac{1}{\varepsilon^2}V_c\left(\frac{z}{\varepsilon}\right)$$

and we obtain a pair eigenfunction/eigenvalue of A^ε by setting

$$\chi^\varepsilon(z) = \frac{1}{\sqrt{\varepsilon}}\chi\left(\frac{z}{\varepsilon}\right) \quad ; \quad E^\varepsilon = \frac{E}{\varepsilon^2}.$$

Remark that the assumption on the eigenfunction given in Assumption 2.2 implies that

$$\forall \beta \in [0,1] \qquad \|z^\beta\, \chi^\varepsilon\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^\beta). \tag{2.1}$$

We shall denote by $X^\varepsilon = span(\chi^\varepsilon)$ the corresponding eigenspace and by Π^ε the orthogonal projector on this eigenspace. Following the physical literature [1, 2], we shall call *subband of energy level E^ε*, the space $L^2(\mathbb{R}^2, X^\varepsilon)$. With an abuse of notation, we shall also denote by Π^ε the orthogonal projector $\mathbb{I}\otimes\Pi^\varepsilon$ of $L^2(\mathbb{R}^3)$ on $L^2(\mathbb{R}^2, X^\varepsilon)$.

The following technical lemma will be used several times :

Lemma 2.3 *Let $V^\varepsilon \in W^{1,\alpha}(\mathbb{R})$ with $\alpha \in [1,+\infty]$. Then there exists a constant $C > 0$ such that*

$$\|[\Pi^\varepsilon, V^\varepsilon]\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq C\,\varepsilon^{1-1/\alpha}\,\|\partial_z V^\varepsilon\|_{L^\alpha(\mathbb{R})},$$

where $[\cdot,\cdot]$ denotes the commutator between the two operators.

Proof. Remarking that

$$[\Pi^\varepsilon, V^\varepsilon] = \Pi^\varepsilon\, V^\varepsilon\,(\mathbb{I}-\Pi^\varepsilon) - (\mathbb{I}-\Pi^\varepsilon)\, V^\varepsilon\,\Pi^\varepsilon$$

and that in this difference the second operator is the adjoint of the first one, one can see that the lemma stems from

$$\|\Pi^\varepsilon\, V^\varepsilon\,(\mathbb{I}-\Pi^\varepsilon)\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq C\,\varepsilon^{1-1/\alpha}\,\|\partial_z V^\varepsilon\|_{L^\alpha(\mathbb{R})}.$$

In order to prove the above estimate, let $U^\varepsilon(z) = V^\varepsilon(z) - V^\varepsilon(0)$. By orthogonality of Π^ε and $\mathbb{I} - \Pi^\varepsilon$, we have clearly

$$\Pi^\varepsilon\, V^\varepsilon\, (\mathbb{I} - \Pi^\varepsilon) = \Pi^\varepsilon\, U^\varepsilon\, (\mathbb{I} - \Pi^\varepsilon).$$

Therefore

$$\|\Pi^\varepsilon\, V^\varepsilon\, (\mathbb{I} - \Pi^\varepsilon)\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \|\Pi^\varepsilon\, U^\varepsilon\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \|\chi^\varepsilon\, U^\varepsilon\|_{L^2(\mathbb{R})},$$

where a Cauchy-Schwarz inequality was used. Besides we have

$$|U^\varepsilon(z)| = \left| \int_0^z \partial_z V^\varepsilon(y)\, dy \right| \leq |z|^{1-1/\alpha}\, \|\partial_z V^\varepsilon\|_{L^\alpha(\mathbb{R})}.$$

Thus we conclude thanks to

$$\begin{aligned} \|\chi^\varepsilon\, U^\varepsilon\|^2_{L^2(\mathbb{R})} &\leq \|\partial_z V^\varepsilon\|^2_{L^\alpha(\mathbb{R})} \left\| z^{1-1/\alpha} \chi^\varepsilon \right\|^2_{L^2(\mathbb{R})} \\ &\leq C\, \varepsilon^{2-2/\alpha}\, \|\partial_z V^\varepsilon\|^2_{L^\alpha(\mathbb{R})}\,, \end{aligned}$$

where we used (2.1). $\square$

2.2 Definitions of the approximate models and main results

We shall assume that the initial wavefunction belongs to the subband of energy level E^ε. Namely:

Assumption 2.4 (well-prepared data) *The initial data ψ_0^ε of the 3D Schrödinger-Poisson problem* (1.1)–(1.3) *satisfies*

$$\psi_0^\varepsilon = \phi_0\, \chi^\varepsilon \in H^1(\mathbb{R}^2, X^\varepsilon).$$

Let us now write the two approximate models for the 3D Schrödinger-Poisson system (1.1)–(1.3).

The 2D surface density model

The 2D surface density model is obtained by coupling a two-dimensional Schrödinger equation and the Poisson equation with a modified Green function. It is given by

$$i\partial_t \phi = -\frac{1}{2}\Delta_x\, \phi + W\phi \tag{2.2}$$

$$W = \frac{1}{4\pi|x|} *_x \left(|\phi|^2\right), \tag{2.3}$$

with the initial data $\phi(0, x) = \phi_0(x) = \langle \psi_0^\varepsilon(x, \cdot)\chi^\varepsilon \rangle$. The unknowns are $\phi(t, x)$, $W(t, x)$ and the surface density $n_s(t, x) = |\phi|^2(t, x)$, where $x \in \mathbb{R}^2$. Remark that

$W(t,x) = V(t,x,0)$, where V is the Coulomb potential generated by the sheet density supported in the plane $z=0$ with a surface density n_s :

$$n(t,x,z) = n_s(t,x)\delta(z) \quad ; \quad V = \frac{1}{4\pi r} * n. \tag{2.4}$$

The 2.5D adiabatic model

The 2.5D adiabatic model is an intermediate model between the fully 3D model and the 2D surface density one. It takes into account the small thickness of the electron gas and consists in coupling a two-dimensional Schrödinger equation and the three-dimensional Poisson equation. The unknowns are $\phi^\varepsilon(t,x)$, $V^\varepsilon(t,x,z)$ and the density $n^\varepsilon(t,x,z)$, where $x \in \mathbb{R}^2$ and $z \in \mathbb{R}$. This system is written

$$i\partial_t \phi^\varepsilon = -\frac{1}{2}\Delta_x \phi^\varepsilon + \left\langle V^\varepsilon |\chi^\varepsilon|^2 \right\rangle \phi^\varepsilon \tag{2.5}$$

$$V^\varepsilon = \frac{1}{4\pi r} * \left(|\phi^\varepsilon|^2 \, |\chi^\varepsilon|^2 \right), \tag{2.6}$$

with the initial data $\phi^\varepsilon(0,x) = \phi_0(x) = \langle \psi_0^\varepsilon(x,\cdot)\chi^\varepsilon \rangle$ and where the function $\chi^\varepsilon(z)$ has been defined in Section 2.1. The population of electrons is described by a pure quantum state which belongs at any time to the subband of energy level E. One can remark that in the 2.5D adiabatic model, the dynamics on the subband is induced by the effective potential $\langle V^\varepsilon |\chi^\varepsilon|^2 \rangle$, which is the potential "modulated" by the wavefunction χ^ε. Moreover, applying formally the standard perturbation theory (see [14]), the transverse Hamiltonian $-\frac{1}{2}\frac{d^2}{dz^2} + V_c^\varepsilon + V^\varepsilon$ admits an eigenvalue $\epsilon(t,x)$ given by

$$\epsilon = \frac{E}{\varepsilon^2} + \left\langle V^\varepsilon |\chi|^2 \right\rangle + \mathcal{O}(\varepsilon^2).$$

Thus, the above 2.5D adiabatic model can be seen –at least formally– as an ε^2-perturbation of the model given by the adiabatic quantum theory [19] (the constant E/ε^2 can be forgotten in (2.5) since it only induces a phase factor).

The main results of the paper, summarized in the three following Theorems, state that the 2.5D adiabatic model is (almost) a second order approximation of the 3D model, while the 2D surface density model is exactly a first order approximation.

Theorem 2.5 *Suppose that Assumptions* 2.2 *and* 2.4 *are satisfied. Then the 3D Schrödinger-Poisson system* (1.1)–(1.3) *and the 2.5D adiabatic model* (2.5), (2.6) *admit unique global weak solutions, respectively denoted by* (ψ^{3D}, V^{3D}) *and by* $(\phi^{2.5D}, V^{2.5D})$*. Moreover for any* T *we have*

$$\|\psi^{3D} - \phi^{2.5D}\chi^\varepsilon \, e^{-itE/\varepsilon^2}\|_{q^*,q,2} = \mathcal{O}(\varepsilon) \qquad \forall q \in [2,\infty), \tag{2.7}$$

$$\|V^{3D} - V^{2.5D}\|_{L^1((0,T),L^\infty(\mathbb{R}^3))} = \mathcal{O}(\varepsilon^{2-\alpha}) \qquad \forall \alpha > 0, \tag{2.8}$$

Furthermore the surface densities defined by $n_s^{3D} = \langle|\psi^{3D}|^2\rangle$ *and* $n_s^{2.5D} = |\phi^{2.5D}|^2$ *satisfy*

$$\|n_s^{3D} - n_s^{2.5D}\|_{L^1((0,T),L^q(\mathbb{R}^2))} = \mathcal{O}(\varepsilon^{2-\alpha}) \qquad \forall\alpha > 0, \qquad \forall q \in [1,\infty). \tag{2.9}$$

Theorem 2.6 *Suppose that Assumptions* 2.2 *and* 2.4 *are satisfied. Then as* $\varepsilon \to 0$ *and for any* $T > 0$ *the solution* ($\phi^{2.5D}$, $n^{2.5D}$, $V^{2.5D}$) *of the 2.5D adiabatic model converges to the unique solution* (ϕ^{2D}, n_s^{2D}, V^{2D}) *of the 2D surface density model* (2.2), (2.4) *in the following sense:*

$$\|\phi^{2.5D} - \phi^{2D}\|_{L^{q^*}((0,T),W^{1,q}(\mathbb{R}^2))} = \mathcal{O}(\varepsilon) \qquad \forall q \in [2,\infty), \tag{2.10}$$

$$\|V^{2.5D} - V^{2D}\|_{L^q((0,T),L^\infty(\mathbb{R}^3))} = \mathcal{O}(\varepsilon) \qquad \forall q \in [1,\infty), \tag{2.11}$$

$$\|n_s^{2.5D} - n_s^{2D}\|_{L^q((0,T),L^\infty(\mathbb{R}^2))} = \mathcal{O}(\varepsilon) \qquad \forall q \in [1,\infty). \tag{2.12}$$

Theorem 2.7 *Suppose that Assumptions* 2.2 *and* 2.4 *are satisfied. If moreover we have*

$$0 < \|x\phi_0\|_{L^2(\mathbb{R}^2)} < +\infty \quad \textit{and} \quad \phi_0 \in H^2(\mathbb{R}^2), \tag{2.13}$$

then for any $T > 0$ *there exists a constant* $C > 0$ *such that the solutions of the 2.5D adiabatic model and the 2D surface density model satisfy*

$$\|(V^{2.5D} - V^{2D})(t,\cdot,0)\|_{L^\infty(\mathbb{R}^2)} + \|(n_s^{2.5D} - n_s^{2D})(t,\cdot)\|_{L^q(\mathbb{R}^2)} \geq C\,\varepsilon, \tag{2.14}$$

for any $t \in [0,T]$, $q \in [1,\infty)$, *where* C *depends on* T *and* q *but not on* ε.

An immediate consequence of these theorems is the

Corollary 2.8 *Under Assumptions* 2.2, 2.4, *the 3D Schrödinger-Poisson system converges as* $\varepsilon \to 0$ *to the 2D surface density model. Moreover, if in addition* (2.13) *is satisfied, we have for any* $T > 0$ *and* $q \in [1,\infty)$,

$$C_1\,\varepsilon \leq \|V^{3D} - V^{2D}\|_{L^1((0,T),L^\infty(\mathbb{R}^3))} + \|n_s^{3D} - n_s^{2D}\|_{L^1((0,T),L^q(\mathbb{R}^2))} \leq C_2\,\varepsilon$$

where the notations of Theorems 2.5 *and* 2.6 *were used.*

3 Estimates for the 3D model

In this section we prove some ε-independent estimates for the 3D Schrödinger-Poisson problem (1.1)–(1.3). We first claim that a straightforward adaptation of the proofs of [4, 13] allows to show that for any initial data

$$\psi_0^\varepsilon \in \mathcal{H} = \{\phi \in H^1(\mathbb{R}^3) \;:\; \sqrt{V_c^\varepsilon}\,\psi \in L^2(\mathbb{R}^3)\}, \tag{3.1}$$

(which may depend on ε) and for an arbitrary $T > 0$, this system admits a unique weak solution ψ^ε, V^ε, such that

$$\psi^\varepsilon \in C([0,T],L^2(\mathbb{R}^3)) \cap L^\infty((0,T),H^1(\mathbb{R}^3)),$$

$$V^\varepsilon \in L^\infty((0,T)\times\mathbb{R}^3) \quad ; \quad \nabla_{x,z}V^\varepsilon \in L^\infty((0,T), L^q(\mathbb{R}^3)) \quad \forall q \in (3/2,\infty).$$

Let us define the kinetic energy along the x direction and along the z one, respectively by:

$$\mathcal{E}^\varepsilon_{kin,x}(t) = \iint_{\mathbb{R}^3} \frac{1}{2}\left|\nabla_x \psi^\varepsilon(t,x,z)\right|^2 dx\,dz, \quad ; \quad \mathcal{E}^\varepsilon_{kin,z}(t) = \iint_{\mathbb{R}^3} \frac{1}{2}\left|\partial_z \psi^\varepsilon(t,x,z)\right|^2 dx\,dz.$$

The selfconsistent potential energy and the external potential energy are then respectively defined by:

$$\mathcal{E}^\varepsilon_{pot}(t) = \iint_{\mathbb{R}^3} \frac{1}{2}\left|\nabla_{x,z}V^\varepsilon\right|^2 dx\,dz \quad ; \quad \mathcal{E}^\varepsilon_{ext}(t) = \iint_{\mathbb{R}^3} V^\varepsilon_c(z)|\psi^\varepsilon(t,x,z)|^2\,dx\,dz$$

and the total energy of the system is

$$\mathcal{E}^\varepsilon_{tot}(t) = \mathcal{E}^\varepsilon_{kin,x}(t) + \mathcal{E}^\varepsilon_{kin,z}(t) + \mathcal{E}^\varepsilon_{pot}(t) + \mathcal{E}^\varepsilon_{ext}(t).$$

The standard energy estimate for the Schrödinger-Poisson system [4] gives the conservation of the total energy:

$$\forall t \geq 0 \qquad \mathcal{E}^\varepsilon_{tot}(t) = \mathcal{E}^\varepsilon_{tot}(0). \tag{3.2}$$

Unfortunately, due to the strong confinement potential V^ε_c, the external energy $\mathcal{E}^\varepsilon_{ext}$ is of order $\mathcal{O}(1/\varepsilon^2)$. Therefore, (3.2) does not provide directly a bound for the kinetic energy (expect for the special case where the initial data is concentrated on the ground state). Nevertheless the Strichartz' estimates of Appendix A enable to obtain some estimates independent of ε, without using the energy conservation. The first step is the following lemma:

Lemma 3.1 *Let $\psi^\varepsilon_0 \in L^2(\mathbb{R}^3)$ and let ψ^ε, V^ε be a solution of* (1.1)–(1.3). *If Assumption* 2.2 *(i) is satisfied, then for any $T > 0$ we have*

$$\forall q \in [2,\infty) \qquad \|\psi^\varepsilon\|_{q^*,q,2} \leq C(\psi_0), \tag{3.3}$$

$$\forall q \in [1,3) \qquad \|V^\varepsilon\|_{L^q((0,T),L^\infty(\mathbb{R}^3))} \leq C(\psi_0), \tag{3.4}$$

where $C(\psi_0)$ denotes a generic constant which depends only on $\|\psi^\varepsilon_0\|_{L^2(\mathbb{R}^3)}$ (and q), and $q^* = 2q/(q-2)$.

Proof. This proof relies on the Strichartz' estimates and on the properties of the Poisson equation studied in Appendices A and B. Let us first recall that the L^2 estimate for the Schrödinger equation gives

$$\forall t \in [0,T] \qquad \|\psi(t)\|_{2,2} \leq \|\psi^\varepsilon_0\|_{L^2(\mathbb{R}^3)}.$$

Besides, from (B.3) and a Hölder inequality, we deduce that

$$\forall q \in (2,\infty) \qquad \left\|\frac{1}{r} * (fg)\right\|_{q,\infty} \leq C\,\|f\|_{q,2}\,\|g\|_{2,2}\,,$$

thus for all $t \in (0,T)$ we have

$$\forall q \in (2,\infty) \qquad \|V^\varepsilon(t)\|_{q,\infty} \le C(\psi_0)\, \|\psi^\varepsilon(t)\|_{q,2}\,.$$

Hence

$$\|V^\varepsilon(t)\,\psi^\varepsilon(t)\|_{2,2} \le \|V^\varepsilon(t)\|_{q,\infty}\, \|\psi^\varepsilon(t)\|_{q^*,2} \le C(\psi_0)\, \|\psi^\varepsilon(t)\|_{q,2}\, \|\psi^\varepsilon(t)\|_{q^*,2}\,.$$

Let q be fixed such that $q \in [4,\infty)$. It is readily seen that

$$\|\psi^\varepsilon\|_{q^*,2} \le \|\psi^\varepsilon\|_{q,2}^{2/(q-2)}\, \|\psi^\varepsilon\|_{2,2}^{(q-4)/(q-2)}\,,$$

which leads to

$$\|V^\varepsilon(t)\,\psi^\varepsilon(t)\|_{2,2} \le C(\psi_0)\, \|\psi^\varepsilon(t)\|_{q,2}^{q^*/2}. \tag{3.5}$$

For any $t \ge 0$, let

$$Y(t) := \|\psi^\varepsilon\|_{L^{q^*}((0,t),L^q_x L^2_z)}.$$

By using (3.5) and a Hölder inegality, we get

$$\|V^\varepsilon\,\psi^\varepsilon\|_{L^1((0,t),L^2(\mathbb{R}^3))} \le C(\psi_0)\, \sqrt{t}\, (Y(t))^{q^*/2}.$$

Consequently the Strichartz' inequality stated in Lemma A.2 gives

$$Y(t) \le C(\psi_0)\left(1 + \sqrt{t}\,(Y(t))^{q^*/2}\right).$$

Since $Y(0) = 0$, this is enough to conclude by continuity that there exists $\widetilde{T}$ and C_0 depending only on $\|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)}$ and q such that $Y(\widetilde{T}) \le C_0$. We deduce (3.3) for $q \ge 4$ by iterating this procedure on the interval $(\widetilde{T}, 2\widetilde{T})$, then on $(2\widetilde{T}, 3\widetilde{T})$, etc. By interpolation, we also deduce that (3.3) holds true for $q \in (2,4)$. To obtain (3.4), it is enough to apply (B.5) with p close to 2 and to use (3.3) with q close to 4. □

From this lemma, one can deduce the main result of this section:

Proposition 3.2 *Assume that the initial data $\psi_0^\varepsilon \in \mathcal{H}$ (defined by (3.1)) satisfies*

$$\|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} + \|\nabla_x \psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} \le C \tag{3.6}$$

and let ψ^ε, V^ε be the solution of (1.1)–(1.3). Then, if Assumption 2.2 (i) is satisfied, we have the following estimates:

$$\forall q \in [2,\infty) \qquad \|\psi^\varepsilon\|_{q^*,q,2} + \|\nabla_x \psi^\varepsilon\|_{q^*,q,2} \le C, \tag{3.7}$$

$$\|V^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^3)} \le C, \tag{3.8}$$

$$\forall q \in (2,\infty) \qquad \|\nabla_{x,z} V^\varepsilon\|_{\infty,q,\infty} \le C. \tag{3.9}$$

Here C denotes a generic constant independent of ε.

Proof. The first remark is that, thanks to (3.6), the estimates (3.3) and (3.4) given in the previous lemma are independent of ε. Differentiating (1.1) with respect to x leads to

$$i\partial_t \nabla_x \psi^\varepsilon = -\frac{1}{2}\Delta_x \nabla_x \psi^\varepsilon + A^\varepsilon \nabla_x \psi^\varepsilon + V^\varepsilon \nabla_x \psi^\varepsilon + \nabla_x V^\varepsilon \psi^\varepsilon. \tag{3.10}$$

From (B.4), we deduce that for all $t \in (0,T)$ we have

$$\begin{aligned} \forall q \in (2,\infty) \qquad \|\nabla_x V^\varepsilon(t)\|_{q,\infty} &\le C\,\|\nabla_x(|\psi^\varepsilon(t)|^2)\|_{2q/(2+q),1} \\ &\le C\,\|\nabla_x \psi^\varepsilon(t)\|_{L^2(\mathbb{R}^3)}\,\|\psi^\varepsilon(t)\|_{q,2}\,. \end{aligned}$$

Hence we get, for any $q \in (2,\infty)$,

$$\|\nabla_x V^\varepsilon\|_{q',q,\infty} \le C\,\|\nabla_x \psi^\varepsilon\|_{2,2,2}\,\|\psi^\varepsilon\|_{q^*,q,2}\,, \tag{3.11}$$

since $\frac{1}{q^*} + \frac{1}{2} = \frac{1}{q'}$. Therefore we have

$$\|\nabla_x V^\varepsilon \psi^\varepsilon\|_{1,2,2} \le \|\nabla_x V^\varepsilon\|_{q',q,\infty}\,\|\psi^\varepsilon\|_{q,q^*,2} \le C\,\|\nabla_x \psi^\varepsilon\|_{2,2,2}\,\|\psi^\varepsilon\|_{q^*,q,2}\,\|\psi^\varepsilon\|_{q,q^*,2}\,.$$

Since for any $q \in (2,\infty)$ we have $(q^*)^* = q$, by using (3.3) we obtain

$$\|\nabla_x V^\varepsilon \psi^\varepsilon\|_{1,2,2} \le C\|\nabla_x \psi^\varepsilon\|_{2,2,2}\,.$$

This inequality, combined with the L^2 estimate for (3.10), gives

$$\begin{aligned} \|\nabla_x \psi^\varepsilon\|_{\infty,2,2} &\le \|\nabla_x \psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} + \|\nabla_x V^\varepsilon \psi^\varepsilon\|_{1,2,2} \\ &\le \|\nabla_x \psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} + C\,\|\nabla_x \psi^\varepsilon\|_{2,2,2}\,, \end{aligned}$$

which leads thanks to a Gronwall argument, to

$$\|\nabla_x \psi^\varepsilon\|_{\infty,2,2} + \|\nabla_x V^\varepsilon \psi^\varepsilon\|_{1,2,2} \le C\,. \tag{3.12}$$

In a second step, we apply the Strichartz' estimate (A.5) to (3.10) and obtain

$$\forall q \in [2,\infty) \qquad \|\nabla_x \psi^\varepsilon\|_{q^*,q,2} \le C\|\nabla_x \psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} + C\|V^\varepsilon \nabla_x \psi^\varepsilon\|_{1,2,2} + C\|\nabla_x V^\varepsilon \psi^\varepsilon\|_{1,2,2}\,.$$

Since (3.4) implies

$$\|V^\varepsilon \nabla_x \psi^\varepsilon\|_{1,2,2} \le \|V^\varepsilon\|_{1,\infty,\infty}\,\|\nabla_x \psi^\varepsilon\|_{\infty,2,2} \le C\|\nabla_x \psi^\varepsilon\|_{\infty,2,2}\,,$$

we deduce the estimate (3.7) from (3.6) and (3.12).

For the last step of this proof, we apply a Sobolev estimate pointwise in time to the function

$$u(t,x) = \|\psi^\varepsilon(t,x,\cdot)\|_{L^2(\mathbb{R})}\,.$$

To this aim, by using the Cauchy-Schwarz inequality, we first get

$$|\nabla_x u(t,x)| \le \left(\int |\nabla_x \psi^\varepsilon(t,x,z)|^2\,dz\right)^{1/2},$$

which yields

$$\|u(t,\cdot)\|_{H^1(\mathbb{R}^2)} \le C\|\psi^\varepsilon\|_{\infty,2,2} + C\|\nabla_x\psi^\varepsilon\|_{\infty,2,2} \le C$$

(apply (3.7) with $q=2$ for the last inequality). By Sobolev embeddings, we have

$$\forall p \in [2,\infty) \qquad \|\psi^\varepsilon\|_{\infty,p,2} = \|u\|_{L_t^\infty L_x^p} \le C, \tag{3.13}$$

which can be rewritten

$$\forall q \in [1,\infty) \qquad \left\||\psi^\varepsilon|^2\right\|_{\infty,q,1} \le C\,.$$

From (B.5) we deduce the $L^\infty((0,T)\times\mathbb{R}^3)$ estimate (3.8). Finally, by combining (3.13) and (3.7), we deduce that

$$\forall q \in (1,2) \qquad \left\|\nabla_x\left(|\psi^\varepsilon|^2\right)\right\|_{\infty,q,1} \le \|\psi^\varepsilon\|_{\infty,2q/(2-q),2}\,\|\nabla_x\psi^\varepsilon\|_{\infty,2,2} \le C,$$

and (3.9) is obtained by applying (B.4). □

We end this section with a useful lemma concerning the linear Schrödinger equation with a strong confining potential. It states that, up to –at least– the first order in ε, the subspace X^ε is stable under the action of the Schrödinger group.

Lemma 3.3 *Let $\psi_0^\varepsilon \in L^2(\mathbb{R}^2, X^\varepsilon)$. Assume that $V^\varepsilon \in L^1((0,T), L^\infty(\mathbb{R}^3))$ and that $\partial_z V^\varepsilon \in L^{r',r,\infty}((0,T)\times(\mathbb{R}^3))$ for some $r\in(2,\infty]$. Then any solution ψ^ε of* (1.1) *satisfies, for all $s\in[2,\infty)$,*

$$\|(\mathbb{I}-\Pi^\varepsilon)\psi^\varepsilon\|_{s^*,s,2} \le C\,\varepsilon\,\|\partial_z V^\varepsilon\|_{r',r,\infty}\,\|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)}\,,$$

where C depends only on $\|V^\varepsilon\|_{1,\infty,\infty}$.

Proof. Thanks to the conservation of the L^2 norm for the Schrödinger equation, a solution ψ^ε of (1.1) satisfies

$$\|\psi^\varepsilon\|_{\infty,2,2} \le \|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)},$$

By using (A.5), we get for any $q\in[2,\infty)$:

$$\|\psi^\varepsilon\|_{q^*,q,2} \le C\|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} + C\|V^\varepsilon\,\psi^\varepsilon\|_{1,2,2} \le C\|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} \tag{3.14}$$

(in this lemma, C is a generic constant depending only on $\|V^\varepsilon\|_{1,\infty,\infty}$).

Denote $\omega^\varepsilon = (\mathbb{I}-\Pi^\varepsilon)\psi^\varepsilon$. The assumption on ψ_0^ε implies $\omega^\varepsilon(0,x,z)=0$ for $(x,z)\in\mathbb{R}^3$. Besides, the operator $\mathbb{I}-\Pi^\varepsilon$ commutes with ∂_t, with Δ_x and with A^ε (since Π^ε is a spectral projector of A^ε). Hence (1.1) gives, after direct calculations:

$$\begin{cases} i\partial_t\omega^\varepsilon = -\dfrac{1}{2}\Delta_x\omega^\varepsilon + A^\varepsilon\omega^\varepsilon + V^\varepsilon\omega^\varepsilon - [\Pi^\varepsilon, V^\varepsilon]\psi^\varepsilon, \\ \omega^\varepsilon(0,x,z) = 0. \end{cases} \tag{3.15}$$

Because of source terms, the L^2 conservation becomes

$$\|\omega^\varepsilon\|_{\infty,2,2} \le C \, \|[\Pi^\varepsilon, V^\varepsilon]\psi^\varepsilon\|_{1,2,2} \, ,$$

thus from (A.5) with $\sigma \in [2,\infty)$ we deduce:

$$\|\omega^\varepsilon\|_{\sigma^*,\sigma,2} \le C \, \|[\Pi^\varepsilon, V^\varepsilon]\psi^\varepsilon\|_{1,2,2} \, . \tag{3.16}$$

Besides Lemma 2.3 yields

$$\|[\Pi^\varepsilon, V^\varepsilon]\psi^\varepsilon(t,x,\cdot)\|_{L^2(\mathbb{R})} \le C\,\varepsilon \, \|\partial_z V^\varepsilon(t,x,\cdot)\|_{L^\infty(\mathbb{R})} \, \|\psi^\varepsilon(t,x,\cdot)\|_{L^2(\mathbb{R})}.$$

Hence

$$\|[\Pi^\varepsilon, V^\varepsilon]\psi^\varepsilon\|_{1,2,2} \le C\,\varepsilon \, \|\partial_z V^\varepsilon\|_{r',r,\infty} \, \|\psi^\varepsilon\|_{r,r^*,2}.$$

An application of (3.14) with $q = r^*$ gives

$$\|[\Pi^\varepsilon, V^\varepsilon]\psi^\varepsilon\|_{1,2,2} \le C\,\varepsilon \, \|\partial_z V^\varepsilon\|_{r',r,\infty} \, \|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} \, .$$

Therefore we deduce the result from this estimate and (3.16). □

4 Existence results for the approximate models

In this section we show that the two approximate models (2.5), (2.6) and (2.2), (2.3) presented in Section 2 are well posed. Let us first remark that the 2.5D adiabatic model can be rewritten as a two-dimensional Schrödinger-Poisson system with a modified Green function. Indeed, denoting $W^\varepsilon(x) = \langle V^\varepsilon | \chi^\varepsilon|^2 \rangle$, (2.5), (2.6) is equivalent to

$$i\partial_t \phi^\varepsilon = -\frac{1}{2}\Delta_x \, \phi^\varepsilon + W^\varepsilon \phi^\varepsilon \tag{4.1}$$

$$W^\varepsilon(x) = G^{2.5D} *_x \left(|\phi^\varepsilon|^2 \right), \tag{4.2}$$

where

$$G^{2.5D}(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{4\pi \left(|x|^2 + (z-z')^2 \right)^{1/2}} \, |\chi^\varepsilon(z')|^2 \, |\chi^\varepsilon(z)|^2 \, dz' \, dz. \tag{4.3}$$

With this formulation, both approximate systems have the same structure, they differ by the kernel of the “Poisson” equation, respectively $G^{2.5D}(x)$ for (4.1), (4.2) and $G^{2D}(x) = \frac{1}{4\pi|x|}$ for (2.2), (2.3). We shall see below that these kernels share the same properties and that their difference is small (see the proof of Theorem 2.6 in Section 4.2).

4.1 A Schrödinger-Poisson system with a general kernel

Let $G^\varepsilon(x)$ be a general convolution kernel such that $G^\varepsilon \in L^1_{loc}(\mathbb{R}^2)$. Consider the system:

$$i\partial_t \phi^\varepsilon = -\frac{1}{2}\Delta_x \phi^\varepsilon + W^\varepsilon \phi^\varepsilon \tag{4.4}$$

$$W^\varepsilon = G^\varepsilon * |\phi^\varepsilon|^2, \tag{4.5}$$

with the initial data $\phi^\varepsilon(0,\cdot) = \phi_0$. In this problem, the dependency of the functions in ε comes from the dependency of G^ε in this parameter. The energy of this system has two terms: the kinetic energy along x and the potential energy respectively defined by

$$\mathcal{E}^\varepsilon_{kin}(t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_x \phi^\varepsilon(t,x)|^2\, dx,$$

$$\mathcal{E}^\varepsilon_{pot}(t) = \frac{1}{2}\int_{\mathbb{R}^2} W^\varepsilon\, n^\varepsilon_s\, dx = \frac{1}{2}\iint_{\mathbb{R}^4} G^\varepsilon(x-x') n^\varepsilon_s(x) n^\varepsilon_s(x')\, dxdx'.$$

By analogy with the function $\frac{1}{|x|}$ (see Lemma B.1), we assume that the kernel G^ε satisfies the following property:

Assumption 4.1 *The kernel G^ε is a nonnegative, even function which belongs to $L^1_{loc}(\mathbb{R}^2)$. Moreover, we assume the following estimates (i) For $f \in L^q(\mathbb{R}^2)$ with $1 < q < 2$, then*

$$\|G^\varepsilon * f\|_{L^{q^\#}(\mathbb{R}^2)} \le C\, \|f\|_{L^q(\mathbb{R}^2)}, \tag{4.6}$$

where $q^\# = \frac{2q}{2-q}$.
(ii) For $f \in L^q(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $2 < q \le +\infty$, the following estimates holds

$$\|G^\varepsilon * f\|_{L^\infty(\mathbb{R}^2)} \le C\, \|f\|^\theta_{L^q(\mathbb{R}^2)} \|f\|^{1-\theta}_{L^1(\mathbb{R}^2)}, \tag{4.7}$$

where $\theta = \frac{q}{2q-2}$. The constants C are assumed independent of ε and f.

Remark. Any kernel of the type $G^\varepsilon(x) = g^\varepsilon(|x|)$, with $g^\varepsilon(|x|)$ satisfying $g^\varepsilon(t) < C/t$, verifies Assumption 4.1.

The following Proposition shows that this system (4.4), (4.5) is well-posed and gives some ε-independent estimates:

Proposition 4.2 *Under Assumption* 4.1 *and for $\phi_0 \in H^1(\mathbb{R}^2)$, the system* (4.4), (4.5) *admits a unique global weak solution. Moreover the total energy of the system is conserved:*

$$\mathcal{E}^\varepsilon_{kin}(t) + \mathcal{E}^\varepsilon_{pot}(t) = \mathcal{E}^\varepsilon_{kin}(0) + \mathcal{E}^\varepsilon_{pot}(0) \tag{4.8}$$

and for any $T > 0$ the following estimates hold, independently of ε:

$$\|\phi^\varepsilon\|_{L^{q^*}((0,T),W^{1,q}(\mathbb{R}^2))} \le C \qquad \forall q \in [2,\infty), \tag{4.9}$$

$$\|W^\varepsilon\|_{L^\infty((0,T),W^{1,q}(\mathbb{R}^2))} \le C \qquad \forall q \in (2,\infty). \tag{4.10}$$

Proof. The local-in-time existence of a unique weak solution is obtained via a standard fixed point procedure and is only sketched here. For more details we refer to [4, 13]. Denoting $W^\varepsilon(\psi) = G^\varepsilon * |\psi|^2$, it is enough to show that the application $\mathcal{F} : \psi \mapsto W^\varepsilon(\psi)\psi$ is locally Lipschitz in $H^1(\mathbb{R}^2)$, uniformly in time. To this aim, we shall make use of the following inequalities obtained by simple arguments like Sobolev embeddings and Cauchy-Schwartz inequalities

$$\|fg\|_{H^1(\mathbb{R}^2)} \leq C\|f\|_{W^{1,4}(\mathbb{R}^2)}\|g\|_{H^1(\mathbb{R}^2)} \quad ; \quad \|fg\|_{W^{1,4/3}(\mathbb{R}^2)} \leq C\|f\|_{H^1(\mathbb{R}^2)}\|g\|_{H^1(\mathbb{R}^2)}. \tag{4.11}$$

Let Φ and Ψ be two functions in $H^1(\mathbb{R}^2)$. We have

$$\|\mathcal{F}(\Psi) - \mathcal{F}(\Phi)\|_{H^1(\mathbb{R}^2)} \leq \|W^\varepsilon(\Psi)\,(\Psi - \Phi)\,\|_{H^1(\mathbb{R}^2)} + \|\,(W^\varepsilon(\Psi) - W^\varepsilon(\Phi))\,\Phi\|_{H^1(\mathbb{R}^2)}.$$

Using the first inequality of (4.11), the r.h.s is controlled by

$$\|W^\varepsilon(\Psi)\|_{W^{1,4}(\mathbb{R}^2)}\|\Psi - \Phi\|_{H^1(\mathbb{R}^2)} + \|W^\varepsilon(\Psi) - W^\varepsilon(\Phi)\|_{W^{1,4}(\mathbb{R}^2)}\|\Phi\|_{H^1(\mathbb{R}^2)}.$$

Besides,

$$\begin{aligned} \|W^\varepsilon(\Psi) - W^\varepsilon(\Phi)\|_{W^{1,4}(\mathbb{R}^2)} &\leq \left\|G^\varepsilon * \left(|\Psi|^2 - |\Phi|^2\right)\right\|_{W^{1,4}(\mathbb{R}^2)} \\ &\leq C\,\left\||\Psi|^2 - |\Phi|^2|\right\|_{W^{1,4/3}(\mathbb{R}^2)} \\ &\leq C\|\Psi - \Phi\|_{H^1(\mathbb{R}^2)}\|\Psi + \Phi\|_{H^1(\mathbb{R}^2)}, \end{aligned}$$

where (4.6) is used as well as the second inequality of (4.11). By noticing that $W^\varepsilon(0) = 0$, we conclude that

$$\begin{aligned} \|\mathcal{F}(\Psi) - \mathcal{F}(\Phi)\|_{H^1(\mathbb{R}^2)} \leq\ & C\|\Psi\|^2_{H^1(\mathbb{R}^2)}\|\Psi - \Phi\|_{H^1(\mathbb{R}^2)} \\ & + C\|\Psi - \Phi\|_{H^1(\mathbb{R}^2)}\|\Psi + \Phi\|_{H^1(\mathbb{R}^2)}\|\Phi\|_{H^1(\mathbb{R}^2)} \end{aligned}$$

which proves that $\mathcal{F}$ is locally Lipschitz on $H^1(\mathbb{R}^2)$.

The energy estimate (4.8) shows that the solution is global in time. It can be obtained in a standard manner by multiplying (4.4) by $\partial_t\overline{\phi}^\varepsilon$, integrating on $\mathbb{R}^2$ and taking the real part. The key point is that the nonlinear term can be written as follows:

$$\begin{aligned} \mathcal{R}e\int_{\mathbb{R}^2} W^\varepsilon\phi^\varepsilon\,\partial_t\overline{\phi}^\varepsilon\,dx &= \int_{\mathbb{R}^2} G^\varepsilon * |\phi^\varepsilon|^2(x)\partial_t|\phi^\varepsilon(x)|^2\,dx \\ &= \frac{1}{4}\frac{d}{dt}\iint_{\mathbb{R}^4} G^\varepsilon(x - x')\,|\phi^\varepsilon(x')|^2\,|\phi^\varepsilon(x)|^2\,dx = \frac{1}{2}\frac{d}{dt}\mathcal{E}^\varepsilon_{pot}(t), \end{aligned}$$

where we have symmetrized the formula by using the properties of G^ε. The proof of (4.9) and (4.10) can be done without any difficulty by an adaptation of Lemma 3.1 and Proposition 3.2. The starting point is the $L^\infty((0,T), H^1(\mathbb{R}^2))$ bound of ϕ^ε given by the energy estimate and the conservation of charge density. Then we use successively Assumption 4.1 and standard Strichartz' estimates in dimension 2 (see for instance [7]). $\square$

The following Proposition shows the Lipschitz dependency of the solution of (4.4), (4.5) with respect to the kernel:

Proposition 4.3 *Let G^ε and $\widetilde{G^\varepsilon}$ satisfying Assumption* 4.1 *and such that $G^\varepsilon - \widetilde{G^\varepsilon} \in L^1(\mathbb{R}^2)$. Let $\phi_0 \in H^1(\mathbb{R}^2)$ and denote respectively by $(\phi^\varepsilon,\ W^\varepsilon)$ and $(\widetilde{\phi^\varepsilon},\ \widetilde{W^\varepsilon})$ the solutions of* (4.4), (4.5) *corresponding to these kernels. Then we have*

$$\|\phi^\varepsilon - \widetilde{\phi^\varepsilon}\|_{L^{q^*}((0,T),W^{1,q}(\mathbb{R}^2))} \leq C\|G^\varepsilon - \widetilde{G^\varepsilon}\|_{L^1(\mathbb{R}^2)} \qquad \forall q \in [2,\infty), \tag{4.12}$$

$$\|W^\varepsilon - \widetilde{W^\varepsilon}\|_{L^q((0,T),L^\infty(\mathbb{R}^2))} \leq C\|G^\varepsilon - \widetilde{G^\varepsilon}\|_{L^1(\mathbb{R}^2)} \qquad \forall q \in [1,\infty), \tag{4.13}$$

where C is independent of ε.

Proof. Let us denote $\eta = \|G^\varepsilon - \widetilde{G^\varepsilon}\|_{L^1(\mathbb{R}^2)}$. For any function $f \in L^p(\mathbb{R}^2)$, $p \in [1,+\infty]$, we have

$$\left\|(G^\varepsilon - \widetilde{G^\varepsilon}) * f\right\|_{L^p(\mathbb{R}^2)} \leq \eta \, \|f\|_{L^p(\mathbb{R}^2)}. \tag{4.14}$$

Setting

$$R^\varepsilon(x) = (G^\varepsilon - \widetilde{G^\varepsilon}) * |\widetilde{\phi^\varepsilon}|^2,$$

we have

$$W^\varepsilon - \widetilde{W^\varepsilon} = G^\varepsilon * \left(|\phi^\varepsilon|^2 - |\widetilde{\phi^\varepsilon}|^2\right) + R^\varepsilon. \tag{4.15}$$

By applying (4.9) and the Sobolev embeddings $W^{1,2}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for all $q \in [2,+\infty)$, and $W^{1,p}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ for all $p > 2$, we have

$$\|\widetilde{\phi^\varepsilon}\|_{L^\infty((0,T),L^q(\mathbb{R}^2))} + \|\widetilde{\phi^\varepsilon}\|_{L^q((0,T),L^\infty(\mathbb{R}^2))} \leq C \qquad \forall q \in [2,\infty). \tag{4.16}$$

Therefore (4.14) yields, for any $q \in [2,\infty)$,

$$\|R^\varepsilon\|_{L^\infty((0,T),L^q(\mathbb{R}^2))} + \|R^\varepsilon\|_{L^q((0,T),L^\infty(\mathbb{R}^2))} \leq C\eta. \tag{4.17}$$

In order to estimate the difference $W^\varepsilon - \widetilde{W^\varepsilon}$, we set $u^\varepsilon := \phi^\varepsilon - \widetilde{\phi^\varepsilon}$. This function solves

$$\begin{cases} i\partial_t u^\varepsilon = -\dfrac{1}{2}\Delta_x u^\varepsilon + W^\varepsilon u^\varepsilon + (W^\varepsilon - \widetilde{W^\varepsilon})\widetilde{\phi^\varepsilon} \\ u^\varepsilon(0,\cdot) \equiv 0. \end{cases} \tag{4.18}$$

Thanks to (4.16) we deduce that for any $p \in (2,\infty]$ and any $t \in [0,T]$

$$\|u^\varepsilon\|_{L^\infty((0,t),L^2(\mathbb{R}^2))} \leq C\|W^\varepsilon - \widetilde{W^\varepsilon}\|_{L^1((0,t),L^p(\mathbb{R}^2))}, \tag{4.19}$$

and, by using (4.10) and Strichartz' estimates in dimension 2 [7], we deduce that for any $s \in [2,\infty)$ and $q \in (2,\infty]$ we have

$$\|u^\varepsilon\|_{L^{s^*}((0,t),L^s(\mathbb{R}^2))} \leq C\|W^\varepsilon - \widetilde{W^\varepsilon}\|_{L^1((0,t),L^q(\mathbb{R}^2))}. \tag{4.20}$$

Let $q \in (2,+\infty)$. By using (4.16) (and the same estimate for ϕ^ε) and (4.19), we obtain

$$\left\| |\phi^\varepsilon|^2 - |\widetilde{\phi}^\varepsilon|^2 \right\|_{L^\infty((0,t),L^s(\mathbb{R}^2))} \le C\|W^\varepsilon - \widetilde{W}^\varepsilon\|_{L^1((0,t),L^q(\mathbb{R}^2))}, \tag{4.21}$$

where $s = \frac{2q}{2+q}$. By (4.6), we deduce

$$\left\| G^\varepsilon * \left(|\phi^\varepsilon|^2 - |\widetilde{\phi}^\varepsilon|^2 \right) \right\|_{L^q(\mathbb{R}^2)} (t) \le C \int_0^t \|W^\varepsilon - \widetilde{W}^\varepsilon\|_{L^q(\mathbb{R}^2)}(\tau)\, d\tau.$$

Consequently (4.15) yields

$$\|W^\varepsilon - \widetilde{W}^\varepsilon\|_{L^q(\mathbb{R}^2)}(t) \le C \int_0^t \|W^\varepsilon - \widetilde{W}^\varepsilon\|_{L^q(\mathbb{R}^2)}(\tau)\, d\tau + \|R^\varepsilon\|_{L^q(\mathbb{R}^2)}(t).$$

Thanks to (4.17), we deduce from a Gronwall argument applied to the above inequality that

$$\|W^\varepsilon - \widetilde{W}^\varepsilon\|_{L^\infty((0,T),L^q(\mathbb{R}^2))} \le C\eta \qquad \forall q \in (2,\infty). \tag{4.22}$$

From this estimate together with (4.20), (4.16) and (4.19), we deduce that for any $r \in (2,\infty)$, $s \in (2, r^*)$, we have

$$\left\| |\phi^\varepsilon|^2 - |\widetilde{\phi}^\varepsilon|^2 \right\|_{L^r((0,T),L^s(\mathbb{R}^2))} + \left\| |\phi^\varepsilon|^2 - |\widetilde{\phi}^\varepsilon|^2 \right\|_{L^\infty((0,T),L^1(\mathbb{R}^2))} < C\eta.$$

which leads to (4.13) in view of (4.15), (4.7) and (4.17).

Let us now improve the estimate on $\phi^\varepsilon - \widetilde{\phi}^\varepsilon$ and show that (4.12) holds. To this aim, we first differentiate (4.18) with respect to x and obtain

$$\begin{cases} i\partial_t v^\varepsilon = -\dfrac{1}{2}\Delta_x v^\varepsilon + W^\varepsilon v^\varepsilon + (\nabla_x W^\varepsilon) u^\varepsilon + (\nabla_x W^\varepsilon - \nabla_x \widetilde{W}^\varepsilon)\widetilde{\phi}^\varepsilon + (W^\varepsilon - \widetilde{W}^\varepsilon)\nabla_x \widetilde{\phi}^\varepsilon \\ v^\varepsilon(0,\cdot) \equiv 0, \end{cases} \tag{4.23}$$

where we have denoted $v^\varepsilon = \nabla_x u^\varepsilon$. By combining (4.9), (4.10), (4.20) and (4.22), we get

$$\|(\nabla_x W^\varepsilon) u^\varepsilon + (W^\varepsilon - \widetilde{W}^\varepsilon)\nabla_x \widetilde{\phi}^\varepsilon\|_{L^1((0,T),L^2(\mathbb{R}^2))} \le C\eta,$$

thus, for any $q \in (2,\infty]$ and $t \in [0,T]$, we have

$$\|v^\varepsilon\|_{L^2(\mathbb{R}^2)}(t) \le C\eta + C\|\nabla_x W^\varepsilon - \nabla_x \widetilde{W}^\varepsilon\|_{L^1((0,t),L^q(\mathbb{R}^2))}. \tag{4.24}$$

Besides, by (4.9) and (4.16) and the Young's inequality (4.14), we get

$$\left\| (G^\varepsilon - \widetilde{G}^\varepsilon) * \nabla_x |\widetilde{\phi}^\varepsilon|^2 \right\|_{L^r((0,t),L^q(\mathbb{R}^2))} \le C\eta \qquad \forall q \in (2,\infty) \quad \forall r \in [1, q^*).$$

Moreover, using (4.16), (4.9) and (4.20), we have for any $s \in (1,2)$

$$\left\| \nabla_x \left(|\phi^\varepsilon|^2 - |\widetilde{\phi}^\varepsilon|^2 \right) \right\|_{L^1((0,t),L^s(\mathbb{R}^2))} \le C\eta + C\|v^\varepsilon\|_{L^1((0,T),L^2(\mathbb{R}^2))}.$$

Thus, writing

$$\nabla_x W^\varepsilon - \nabla_x \widetilde{W}^\varepsilon = G^\varepsilon * \nabla_x \left(|\phi^\varepsilon|^2 - |\widetilde{\phi}^\varepsilon|^2 \right) + (G^\varepsilon - \widetilde{G}^\varepsilon) * \nabla_x |\widetilde{\phi}^\varepsilon|^2 \tag{4.25}$$

and using (4.6), we deduce that for any $q \in (2, \infty)$

$$\left\| \nabla_x W^\varepsilon - \nabla_x \widetilde{W}^\varepsilon \right\|_{L^1((0,t),L^q(\mathbb{R}^2))} \leq C\eta + C\|v^\varepsilon\|_{L^1((0,t),L^2(\mathbb{R}^2))}.$$

Inserting this inequality in (4.24) leads through a Gronwall argument to

$$\|v^\varepsilon\|_{L^\infty((0,T),L^2(\mathbb{R}^2))} \leq C\eta.$$

Going back to (4.23), it is readily seen from the above two estimates and from Proposition 4.2 that

$$\|i\partial_t v^\varepsilon + \frac{1}{2}\Delta v^\varepsilon\|_{L^1((0,T),L^2(\mathbb{R}^2))} \leq C\eta$$

which leads to (4.12) through a Strichartz' estimate.

□

In Section 6, in order to get the estimates from below of Theorem 2.7, we will need to deal with strong solutions.

Lemma 4.4 *Under Assumption* 4.1, *let* $\phi_0 \in H^2(\mathbb{R}^2)$*. Then for any* $T > 0$ *the solution* ϕ^ε *of* (4.4), (4.5) *belongs to* $L^\infty((0,T), H^2(\mathbb{R}^2))$ *and its norm is bounded independently of* ε*.*

Proof. Denote $u^\varepsilon = \Delta_x \phi^\varepsilon$. By differentiating twice (4.4) with respect to x, we get

$$i\partial_t u^\varepsilon = -\frac{1}{2}\Delta_x u^\varepsilon + W^\varepsilon u^\varepsilon + 2\nabla_x W^\varepsilon \cdot \nabla_x \phi^\varepsilon + \Delta_x W^\varepsilon \phi^\varepsilon.$$

The source term in this Schrödinger equation on u^ε writes

$$2\nabla_x W^\varepsilon \cdot \nabla_x \phi^\varepsilon + \phi^\varepsilon G^\varepsilon * \left(2|\nabla_x \phi^\varepsilon|^2\right) + 2\phi^\varepsilon\, \mathcal{R}e\, G^\varepsilon * \left(\overline{\phi^\varepsilon} u^\varepsilon\right).$$

The first term $\nabla_x W^\varepsilon \cdot \nabla_x \phi^\varepsilon$ can be estimated thanks to (4.9) and (4.10):

$$\|\nabla_x W^\varepsilon \cdot \nabla_x \phi^\varepsilon\|_{L^1((0,t),L^2(\mathbb{R}^2))} \leq \|\nabla_x W^\varepsilon\|_{L^{4/3}((0,t),L^4(\mathbb{R}^2))}\, \|\nabla_x \phi^\varepsilon\|_{L^4((0,t),L^4(\mathbb{R}^2))} \leq C.$$

The second term can be estimated thanks to (4.6):

$$\begin{aligned} &\left\|\phi^\varepsilon G^\varepsilon * \left(2|\nabla_x \phi^\varepsilon|^2\right)\right\|_{L^1((0,t),L^2(\mathbb{R}^2))} \\ &\qquad \leq \|\phi^\varepsilon\|_{L^{3/2}((0,t),L^3(\mathbb{R}^2))}\, \left\|G^\varepsilon * \left(2|\nabla_x \phi^\varepsilon|^2\right)\right\|_{L^3((0,t),L^6(\mathbb{R}^2))} \\ &\qquad \leq C\|\phi^\varepsilon\|_{L^{3/2}((0,t),L^3(\mathbb{R}^2))}\, \|\nabla_x \phi^\varepsilon\|^2_{L^6((0,t),L^3(\mathbb{R}^2))} \leq C. \end{aligned}$$

To treat the third term, we also apply (4.6), (4.9) and (4.10):

$$
\begin{aligned}
&\|\phi^\varepsilon G^\varepsilon * (2\phi^\varepsilon u^\varepsilon)\|_{L^1((0,t),L^2(\mathbb{R}^2))} \\
&\qquad \le C\|\phi^\varepsilon\|_{L^\infty((0,t),L^3(\mathbb{R}^2))}\ \|G^\varepsilon * (2\phi^\varepsilon u^\varepsilon)\|_{L^1((0,t),L^6(\mathbb{R}^2))} \\
&\qquad \le C\|\phi^\varepsilon\|_{L^\infty((0,t),L^3(\mathbb{R}^2))}\ \|\phi^\varepsilon\|_{L^\infty((0,t),L^6(\mathbb{R}^2)}\ \|u^\varepsilon\|_{L^1((0,t),L^2(\mathbb{R}^2))} \\
&\qquad \le C\ \|u^\varepsilon\|_{L^1((0,t),L^2(\mathbb{R}^2))}\,.
\end{aligned}
$$

Hence, for any $t \le T$,

$$
\|u^\varepsilon(t)\|_{L^2(\mathbb{R}^2)} \le C + C\int_0^t \|u^\varepsilon(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau
$$

which leads to the result thanks to a Gronwall argument. □

4.2 Application: proof of Theorem 2.6

Thanks to Lemma B.1 given in the Appendix, the kernel

$$
G^{2D}(x) = \frac{1}{4\pi|x|}
$$

of the 2D surface density model (2.2)-(2.3) satisfies clearly Assumption 4.1. Moreover, by using Lemma B.2 and the fact that $\int_{\mathbb{R}} |\chi^\varepsilon|^2 dz = 1$, it is readily seen that the kernel of the 2.5D adiabatic model given by

$$
G^{2.5D}(x) = \iint_{\mathbb{R}^2} \frac{1}{4\pi\,(|x|^2 + (z-z')^2)^{1/2}}\, |\chi^\varepsilon(z')|^2\, |\chi^\varepsilon(z)|^2\, dz'\, dz
$$

also satisfies Assumption 4.1. Therefore an application of Proposition 4.2 gives the existence of unique weak solutions and estimates independent of ε for the two approximate models. The first parts of Theorems 2.5 and of Theorem 2.6 are thus proved.

To conclude the proof of Theorem 2.6, it suffices to apply Proposition 4.3. Indeed, setting

$$
\begin{aligned}
H^\varepsilon(x) &= \frac{1}{4\pi|x|} - G^{2.5D}(x) \\
&= \frac{1}{4\pi|x|} - \iint_{\mathbb{R}^2} \frac{1}{4\pi\,(|x|^2 + (z-z')^2)^{1/2}}\, |\chi^\varepsilon(z')|^2\, |\chi^\varepsilon(z)|^2\, dz\, dz' \\
&= \frac{1}{4\pi} \iint_{\mathbb{R}^2} \int_0^{\varepsilon|z-z'|} \frac{\xi}{(|x|^2+\xi^2)^{3/2}}\, |\chi(z)|^2\, |\chi(z')|^2\, d\xi\, dz\, dz',
\end{aligned}
$$

and noticing that

$$\int_{\mathbb{R}^2} \frac{\xi}{(|x|^2+\xi^2)^{3/2}}\,dx = 2\pi, \quad \text{for } \xi > 0,$$

we deduce from (2.1) that

$$\|H^\varepsilon\|_{L^1(\mathbb{R}^2)} = \frac{\varepsilon}{2}\iint_{\mathbb{R}^2} |z-z'|\,|\chi(z)|^2\,|\chi(z')|^2\,dz\,dz' = C\varepsilon.$$

This leads to (2.10), from which we deduce (2.12). In order to prove (2.11), we write

$$V^{2.5D} - V^{2D} = \frac{1}{4\pi r} *_x \left(n_s^{2.5D} - n_s^{2D}\right) + \widetilde{H^\varepsilon} *_x n_s^{2.5D},$$

where

$$\widetilde{H^\varepsilon}(x,z) = -\frac{1}{4\pi r} + \frac{1}{4\pi r} *_z |\chi^\varepsilon|^2. \tag{4.26}$$

It is then enough to remark that

$$\begin{aligned}\left\|\frac{1}{4\pi r} *_x \left(n_s^{2.5D} - n_s^{2D}\right)\right\|_{L^q((0,T),L^\infty(\mathbb{R}^3))} &\leq \left\|\frac{1}{4\pi|x|} *_x \left(n_s^{2.5D} - n_s^{2D}\right)\right\|_{L^q((0,T),L^\infty(\mathbb{R}^2))} \\ &\leq C\varepsilon\end{aligned}$$

and that

$$\widetilde{H^\varepsilon}(x,z) = \int_{\mathbb{R}}\int_{(z-z')}^{z} \frac{\xi}{(|x|^2+\xi^2)^{3/2}}\,|\chi^\varepsilon(z')|^2\,d\xi\,dz',$$

which implies

$$\begin{aligned}&|\widetilde{H^\varepsilon} *_x n_s^{2.5D}|(t,x,z) \\ &\quad\leq \|n_s^{2.5D}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}}\int_{min(z,z-z')}^{max(z,z-z')}\int_{\mathbb{R}^2} \frac{|\xi|}{(|x|^2+\xi^2)^{3/2}}\,|\chi^\varepsilon(z')|^2\,dxd\xi dz', \\ &\quad= 2\pi\|n_s^{2.5D}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}} |z'|\,|\chi^\varepsilon(z')|^2\,dz' = C\varepsilon\|n_s^{2.5D}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)},\end{aligned}$$

and the right-hand side is an $O(\varepsilon)$ is view of (2.12).

5 The 2.5D adiabatic model is a second order approximation

In this section we end the proof of Theorem 2.5, initiated in Section 4.2. Consider the solution ψ^{3D}, V^{3D} of (1.1)–(1.3) with the initial data $\psi_0^\varepsilon = \phi_0\,\chi^\varepsilon$ and the solution $\phi^{2.5D}$, $V^{2.5D}$ of (2.5), (2.6), corresponding to the initial data ϕ_0. Assumption 2.4 leads in particular to the uniform in ε estimate

$$\|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} + \|\nabla_x\psi_0^\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C.$$

Proposition 3.2 then implies the following uniform bounds:

$$\|V^{3D}\|_{L^\infty((0,T)\times\mathbb{R}^3)} + \|\nabla_{x,z}V^{3D}\|_{\infty,q,\infty} \le C, \qquad 2 < q < \infty, \tag{5.1}$$

$$\|\psi^{3D}\|_{q^*,q,2} + \|\nabla_x \psi^{3D}\|_{q^*,q,2} \le C \qquad 2 \le q < \infty. \tag{5.2}$$

Furthermore Lemma 3.3 implies

$$\left\|(\mathbb{I} - \Pi^\varepsilon)\,\psi^{3D}\right\|_{q^*,q,2} = \mathcal{O}(\varepsilon), \qquad 2 \le q < \infty. \tag{5.3}$$

We start by proving (2.8). To this aim, we write

$$\begin{aligned} V^{3D} - V^{2.5D} &= \frac{1}{4\pi r} * \left(n^{3D} - n^{2.5D}\right) \\ &= \frac{1}{4\pi r} * \left(|\Pi^\varepsilon \psi^{3D}|^2 - |\chi^\varepsilon \phi^{2.5D}|^2\right) + R_a^\varepsilon + R_b^\varepsilon, \end{aligned} \tag{5.4}$$

where the remainder terms are

$$R_a^\varepsilon = \frac{1}{4\pi r} * |\,(\mathbb{I} - \Pi^\varepsilon)\,\psi^{3D}|^2 \quad ; \quad R_b^\varepsilon = \frac{2}{4\pi r} * \mathcal{R}e\left(\overline{\Pi^\varepsilon \psi^{3D}}\,(\mathbb{I} - \Pi^\varepsilon)\,\psi^{3D}\right).$$

Estimating the remainders R_a^ε and R_b^ε. On the one hand, estimates (5.3), (B.3) and (B.5) lead to

$$\|R_a^\varepsilon\|_{1,q,\infty} \le C\varepsilon^2 \qquad \forall q \in (2,\infty]. \tag{5.5}$$

On the other hand, by orthogonality we have $\langle \overline{\Pi^\varepsilon \psi^{3D}}\,(\mathbb{I} - \Pi^\varepsilon)\,\psi^{3D}\rangle = 0$. Consequently (B.9) implies for any $q \in (2,\infty)$ and pointwise in time

$$\|R_b^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \le C\,\|z\chi^\varepsilon\|_{L^2(\mathbb{R})}^{1-2/q}\,\left\|\psi^{3D}\right\|_{2q,2}\left\|(\mathbb{I} - \Pi^\varepsilon)\,\psi^{3D}\right\|_{2q,2}.$$

Besides, we deduce from (5.2) and the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ that

$$\left\|\psi^{3D}\right\|_{\infty,q,2} \le \left\|\psi^{3D}\right\|_{L^\infty((0,T),H^1(\mathbb{R}^2,L^2(\mathbb{R})))} \le C. \tag{5.6}$$

Moreover, by (2.1) we have $\|z\chi^\varepsilon\|_{L^2(\mathbb{R})}^{1-2/q} = \mathcal{O}(\varepsilon^{1-2/q})$, therefore

$$\|R_b^\varepsilon\|_{L^\infty(\mathbb{R}^3)}\,(t) \le C\varepsilon^{1-2/q}\left\|(\mathbb{I} - \Pi^\varepsilon)\,\psi^{3D}\right\|_{2q,2}(t).$$

Similarly, by (B.8), we have for any $\alpha \in (0,1)$ and $q \in [2,\infty)$

$$\begin{aligned} \|R_b^\varepsilon\|_{q,\infty}\,(t) &\le C\,\|z\chi^\varepsilon\|_{L^2(\mathbb{R})}^{1-\alpha}\left\|\psi^{3D}\right\|_{\frac{4q}{2+\alpha q},2}(t)\left\|(\mathbb{I} - \Pi^\varepsilon)\,\psi^{3D}\right\|_{\frac{4q}{2+\alpha q},2}(t) \\ &\le C\varepsilon^{1-\alpha}\left\|(\mathbb{I} - \Pi^\varepsilon)\,\psi^{3D}\right\|_{\frac{4q}{2+\alpha q},2}(t) \end{aligned}$$

By (5.3), we finally get

$$\forall \alpha \in (0,1) \quad \forall q \in [2,\infty] \qquad \|R_b^\varepsilon\|_{1,q,\infty} \le C\varepsilon^{2-\alpha}, \tag{5.7}$$

where the constant C depends only on α.

Estimating the first term in the r.h.s. of (5.4). We shall estimate the difference

$$w^\varepsilon := \Pi^\varepsilon \psi^{3D} - \chi^\varepsilon \phi^{2.5D} e^{-iE^\varepsilon t}.$$

To this aim, we notice that

$$\begin{cases} i\partial_t w^\varepsilon = -\dfrac{1}{2}\Delta_x w^\varepsilon + A^\varepsilon w^\varepsilon + \langle V^{3D}|\chi^\varepsilon|^2\rangle w^\varepsilon + f^\varepsilon + g^\varepsilon, \\ \omega^\varepsilon(0,x,z) = 0, \end{cases} \tag{5.8}$$

where

$$f^\varepsilon = \langle \left(V^{3D} - V^{2.5D}\right)|\chi^\varepsilon|^2\rangle \chi^\varepsilon \phi^{2.5D} e^{-iE^\varepsilon t} \quad ; \quad g^\varepsilon = \Pi^\varepsilon V^{3D}(\mathbb{I} - \Pi^\varepsilon)\psi^{3D}.$$

Standard L^2 estimates for a Schrödinger equation with a source term then imply

$$\|w^\varepsilon\|_{\infty,2,2} \le \|f^\varepsilon\|_{1,2,2} + \|g^\varepsilon\|_{1,2,2}.$$

Remarking that

$$\Pi^\varepsilon V^{3D}(\mathbb{I} - \Pi^\varepsilon) = \Pi^\varepsilon[\Pi^\varepsilon, V^{3D}](\mathbb{I} - \Pi^\varepsilon),$$

we deduce from Lemma 2.3, (5.1) and (5.3) that

$$\|g^\varepsilon\|_{1,2,2} \le C\varepsilon \|\partial_z V^{3D}\|_{4/3,4,\infty} \|(\mathbb{I} - \Pi^\varepsilon)\psi^{3D}\|_{4,4,2} = \mathcal{O}(\varepsilon^2).$$

Besides, in the same spirit as for the proof of (5.6), applying (4.9) and standard Sobolev embeddings, we get

$$\left\|\phi^{2.5D}\right\|_{L^\infty((0,T),L^q(\mathbb{R}^2))} \le C \qquad \forall q \in [2,\infty). \tag{5.9}$$

Therefore it can be easily seen that for any $q \in (2,\infty]$ we have

$$\|f^\varepsilon\|_{1,2,2} \le C\|V^{3D} - V^{2.5D}\|_{1,q,\infty}$$

and we finally obtain

$$\|w^\varepsilon\|_{\infty,2,2} \le C\|V^{3D} - V^{2.5D}\|_{1,q,\infty} + \mathcal{O}(\varepsilon^2). \tag{5.10}$$

Applying the Strichartz' inequality (A.5) to (5.8) after having noticed the estimate (5.1), we obtain for any $q \in (2,\infty]$, $s \in [2,\infty)$,

$$\|w^\varepsilon\|_{s^*,s,2} \le C\|V^{3D} - V^{2.5D}\|_{1,q,\infty} + \mathcal{O}(\varepsilon^2). \tag{5.11}$$

This gives the following estimate for the first term of the right-hand side of (5.4), for any $q \in (2,\infty)$:

$$\begin{aligned} \left\||\Pi^\varepsilon\psi^{3D}|^2 - |\chi^\varepsilon\phi^{2.5D}|^2\right\|_{\frac{2q}{2+q},1}(t) &\le \left(\|\psi^{3D}\|_{\infty,q,2} + \|\phi^{2.5D}\|_{L^\infty((0,t),L^q(\mathbb{R}^2))}\right)\|w^\varepsilon\|_{\infty,2,2} \\ &\le C\int_0^t \|V^{3D} - V^{2.5D}\|_{q,\infty}(\tau)\,d\tau + \mathcal{O}(\varepsilon^2), \end{aligned}$$

where we used (5.6), (5.9) and (5.10).

End of the proof. By applying (B.3) we deduce

$$\left\| \frac{1}{r} * \left(|\Pi^\varepsilon \psi^{3D}|^2 - |\chi^\varepsilon \phi^{2.5D}|^2 \right) \right\|_{q,\infty} (t) \leq C \int_0^t \|V^{3D} - V^{2.5D}\|_{q,\infty}(\tau)\, d\tau + \mathcal{O}(\varepsilon^2),$$

where $q \in (2,\infty)$. Consequently (5.4) yields

$$\begin{aligned} \|(V^{3D} - V^{2.5D})\|_{q,\infty}(t) \leq \ & C \int_0^t \|(V^{3D} - V^{2.5D})\|_{q,\infty}(\tau)\, d\tau \\ & + \|R_a^\varepsilon\|_{q,\infty}(t) + \|R_b^\varepsilon\|_{q,\infty}(t) + O(\varepsilon^2). \end{aligned}$$

Recalling the estimates (5.5) and (5.7) for the remainders, a Gronwall argument leads to the following bound

$$\|(V^{3D} - V^{2.5D})\|_{\infty,q,\infty} \leq C\varepsilon^{2-\alpha} \qquad \forall q \in (2,\infty), \quad \forall \alpha \in (0,1).$$

To conclude the proof, we insert this estimate into (5.11) and obtain

$$\|w^\varepsilon\|_{s^*,s,2} \leq C\varepsilon^{2-\alpha}, \qquad \forall s \in [2,\infty), \quad \forall \alpha \in (0,1).$$

Then we remark that we have now, for any $q \in [2,\infty)$ and $s < q^*$

$$\left\| |\Pi^\varepsilon \psi^{3D}|^2 - |\chi^\varepsilon \phi^{2.5D}|^2 \right\|_{s,q,1} \leq C\varepsilon^{2-\alpha}, \qquad \forall \alpha \in (0,1)$$

and we apply (B.5). By using again (5.4), (5.5) and (5.7), we find (2.8).

In order to prove (2.7), we simply remark that

$$\|\psi^{3D} - \phi^{2.5D}\chi^\varepsilon\, e^{-itE/\varepsilon^2}\|_{q^*,q,2} \leq \|w_\varepsilon\|_{q^*,q,2} + \|(\mathbb{I} - \Pi^\varepsilon)\psi^{3D}\|_{q^*,q,2}$$

then use (5.3) and (5.11). To prove (2.9), we remark that

$$n_s^{3D} - n_s^{2.5D} = |\Pi^\varepsilon \psi^{3D}|^2 - |\chi^\varepsilon \phi^{2.5D}|^2 + |\, (\mathbb{I} - \Pi^\varepsilon)\, \psi^{3D}|^2.$$

6 The 2D surface density model is a first order approximation

In this section we prove Theorem 2.7, which gives estimates from below, showing that the accuracy of the limit model is exactly $\mathcal{O}(\varepsilon)$. We denote respectively by $\phi^{2.5D}$, $V^{2.5D}$ and by ϕ^{2D}, V^{2D} the solutions of (2.5), (2.6) and (2.2), (2.4). For notational simplicity, we denote

$$V_0^{2.5D}(t,x) = V^{2.5D}(t,x,0) \quad ; \quad V_0^{2D} = V^{2D}(t,x,0).$$

Since we assume that the initial data ϕ_0 belongs to $H^2(\mathbb{R}^2)$, an application of Lemma 4.4 gives

$$\|\phi^{2.5D}\|_{L^\infty((0,T),H^2(\mathbb{R}^2))} + \|\phi^{2D}\|_{L^\infty((0,T),H^2(\mathbb{R}^2))} \leq C. \tag{6.1}$$

Moreover, with (2.10) and the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^{2q}(\mathbb{R}^2)$, we obtain

$$\|n_s^{2.5D} - n_s^{2D}\|_{L^\infty((0,T),L^q(\mathbb{R}^2))} \le C\varepsilon \qquad \forall q \in [1,\infty). \tag{6.2}$$

Now we recall that

$$V_0^{2.5D} - V_0^{2D} = \frac{1}{4\pi|x|} *_x \left(n_s^{2.5D} - n_s^{2D}\right) + \widetilde{H^\varepsilon}(\cdot,0) *_x |\phi^{2.5D}|^2,$$

where $\widetilde{H^\varepsilon}$ is defined in (4.26). Hence, pointwise in time, we get

$$\begin{aligned}\|V_0^{2.5D} - V_0^{2D}\|_{L^\infty(\mathbb{R}^2)} + \left\|\frac{1}{4\pi|x|} *_x \left(n_s^{2.5D} - n_s^{2D}\right)\right\|_{L^\infty(\mathbb{R}^2)} \\ \ge \left\|\widetilde{H^\varepsilon}(\cdot,0) *_x |\phi^{2.5D}|^2\right\|_{L^\infty(\mathbb{R}^2)}.\end{aligned} \tag{6.3}$$

Besides, a straightforward calculation leads to

$$i\partial_t(x\phi^{2.5D}) = -\frac{1}{2}\Delta_x(x\phi^{2.5D}) + V^{2.5D}(x\phi^{2.5D}) + \nabla_x\phi^{2.5D},$$

thus

$$\|x\phi^{2.5D}\|_{L^\infty((0,T),L^2(\mathbb{R}^2))} \le \|x\phi_0\|_{L^2(\mathbb{R}^2)} + \|\nabla_x\phi^{2.5D}\|_{L^1((0,T),L^2(\mathbb{R}^2))} \le C,$$

where we used (4.9). Now let us denote for $R > 0$, $\mathcal{B}_R = \{x \in \mathbb{R}^2, |x| < R\}$. We have

$$\begin{aligned}\|\phi^{2.5D}\|^2_{L^\infty((0,T),L^2(\mathcal{B}_R))} &\ge \|\phi_0\|^2_{L^2(\mathbb{R}^2))} - \frac{1}{R^2}\|x\phi^{2.5D}\|^2_{L^\infty((0,T),L^2(\mathbb{R}^2))} \\ &\ge \|\phi_0\|^2_{L^2(\mathbb{R}^2))} - \frac{C}{R^2}.\end{aligned}$$

Since by assumption we have $\|\phi_0\|_{L^2(\mathbb{R}^2))} = 2\eta > 0$, by choosing R large enough we have

$$\|\phi^{2.5D}\|_{L^\infty((0,T),L^2(\mathcal{B}_R))} > \eta,$$

then

$$\forall t \in [0,T] \qquad \max_{\mathcal{B}_R} |\phi^{2.5D}(t,\cdot)|^2 > \frac{\eta^2}{\pi R^2}.$$

By using (6.1) and the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow C^{0,1/2}(\mathbb{R}^2)$, we deduce finally that there exists $r_0 > 0$, $\alpha > 0$ and $x_0(t) \in \mathbb{R}^2$ defined almost everywhere such that, for a.e. $t \in [0,T]$, we have

$$|\phi^{2.5D}|^2(t,x) > \alpha, \qquad \forall x \in \mathbb{R}^2, \ \text{ such that } |x - x_0(t)| < r_0. \tag{6.4}$$

For $t \in [0,T]$, we have

$$
\begin{aligned}
&\left|\widetilde{H^\varepsilon}(\cdot,0) *_x |\phi^{2.5D}|^2\right|(x_0(t)) \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^{\varepsilon|z'|} \frac{\xi}{(|x'|^2+\xi^2)^{3/2}} |\chi(z')|^2 |\phi^{2.5D}(x_0(t)-x')|^2 \, d\xi dz' dx' \\
&\geq 2\pi\alpha \int_{\mathbb{R}}\int_0^{\varepsilon|z'|}\int_{r=0}^{r_0} \frac{r\xi}{(r^2+\xi^2)^{3/2}} |\chi(z')|^2 \, dr d\xi dz' \\
&= 2\pi\alpha \int_{\mathbb{R}} \varepsilon|z'| \left(1 - \frac{\varepsilon|z'|}{r_0^2 + (r_0^2+\varepsilon^2|z'|^2)^{1/2}}\right) |\chi(z')|^2 \, dz' \\
&\geq C_1\varepsilon - C_2\varepsilon^2 \geq C_0\varepsilon,
\end{aligned}
$$

where $C_0 > 0$ and ε is small enough. Therefore, by applying (6.3) and using (B.2), we have for $t \in [0,T]$,

$$
\|V_0^{2.5D} - V_0^{2D}\|_{L^\infty(\mathbb{R}^2)} + \left\|n_s^{2.5D} - n_s^{2D}\right\|_{L^q(\mathbb{R}^2)}^{\theta} \left\|n_s^{2.5D} - n_s^{2D}\right\|_{L^1(\mathbb{R}^2)}^{1-\theta} \geq C\varepsilon,
$$

with any $2 < q < \infty$ and $\theta = \frac{q}{2q-2}$. Bounding $\left\|n_s^{2.5D} - n_s^{2D}\right\|_{L^1(\mathbb{R}^2)}$ by $C\varepsilon$ in view of (6.2), one deduces for any $q \in (2,\infty)$

$$
\frac{\|V_0^{2.5D} - V_0^{2D}\|_{L^\infty(\mathbb{R}^2)}}{\varepsilon} + \left(\frac{\left\|n_s^{2.5D} - n_s^{2D}\right\|_{L^q(\mathbb{R}^2)}}{\varepsilon}\right)^{\theta} \geq C_0'.
$$

Proceeding analogously, we obtain

$$
\frac{\|V_0^{2.5D} - V_0^{2D}\|_{L^\infty(\mathbb{R}^2)}}{\varepsilon} + \left(\frac{\left\|n_s^{2.5D} - n_s^{2D}\right\|_{L^1(\mathbb{R}^2)}}{\varepsilon}\right)^{1-\theta} \geq C_0'.
$$

Consequently, we deduce that

$$
\|V_0^{2.5D} - V_0^{2D}\|_{L^\infty(\mathbb{R}^2)} + \left\|n_s^{2.5D} - n_s^{2D}\right\|_{L^q(\mathbb{R}^2)} \geq C\varepsilon, \quad \forall q \in (2,+\infty) \tag{6.5}
$$

and

$$
\|V_0^{2.5D} - V_0^{2D}\|_{L^\infty(\mathbb{R}^2)} + \left\|n_s^{2.5D} - n_s^{2D}\right\|_{L^1(\mathbb{R}^2)} \geq C\varepsilon.
$$

The last inequality implies by a simple interpolation argument that (6.5) actually holds for $q \in [1,+\infty)$, which finishes the proof.

Appendix

A Strichartz' estimates in $L^{q^*}_t L^q_x L^2_z$

For any $q \in [2,\infty)$ we recall the notation $q^* = 2q/(q-2)$: in the usual terminology for the Strichartz estimates, the pair (q^*, q) is said to be admissible. The space $L^{q^*}_t L^q_x L^2_z$ was defined in Section 2. Let us first state an extension of the standard Strichartz estimate for Schrödinger equations on $\mathbb{R}^2$ with values in a Hilbert space [6, 7, 11, 20]:

Lemma A.1 *Let $T > 0$ and $\mathcal{H}$ be a separable Hilbert space. For $\psi_0 \in L^2(\mathbb{R}^2, \mathcal{H})$ and $g \in L^1((0,T), L^2(\mathbb{R}^2, \mathcal{H}))$, we consider the solution $\psi(t,x) \in L^\infty((0,T), L^2(\mathbb{R}^2, \mathcal{H}))$ of*

$$\left\{ \begin{array}{l} i\partial_t \psi = -\dfrac{1}{2}\Delta_x \psi + g \\ \psi(0,\cdot) = \psi_0 \,. \end{array} \right. \tag{A.1}$$

Then for any $q \in [2,\infty)$, the function ψ belongs to $L^{q^*}((0,T), L^q(\mathbb{R}^2, \mathcal{H}))$ and satisfies

$$\|\psi\|_{L^{q^*}((0,T),L^q(\mathbb{R}^2,\mathcal{H}))} \le C\|\psi_0\|_{L^2(\mathbb{R}^2,\mathcal{H})} + C\|g\|_{L^1((0,T),L^2(\mathbb{R}^2,\mathcal{H}))}, \tag{A.2}$$

where $C > 0$ denotes a constant.

Proof. Let $(\cdot,\cdot)_{\mathcal{H}}$ denote the scalar product on $\mathcal{H}$ and let $(\chi_p)_{p\in\mathbb{N}^*}$ be a Hilbertian basis of $\mathcal{H}$. We shall make use of the Strichartz estimate for mixed quantum states proved in [5]. For this, let us introduce the following functional space:

$$\widetilde{L^q}(\mathbb{R}^2,\mathcal{H}) = \left\{ \psi \in L^q(\mathbb{R}^2,\mathcal{H}) \;:\quad \|\psi\|^2_{\widetilde{L^q}(\mathbb{R}^2,\mathcal{H})} = \sum_{p\ge 1} \|\psi_p\|^2_{L^q(\mathbb{R}^2)} < +\infty \right\},$$

where we have denoted $\psi_p = (\psi, \chi_p)_{\mathcal{H}}$ (remark that this functional space *a priori* depends on the choice of the Hilbertian basis χ_p). This space is continuously embedded in $L^q(\mathbb{R}^2, \mathcal{H})$; indeed we have

$$\|\psi\|_{L^q(\mathbb{R}^2,\mathcal{H})} = \left\| \sum_{p\ge1} |\psi_p|^2 \right\|^{1/2}_{L^{q/2}(\mathbb{R}^2)} \le \left(\sum_{p\ge1} \left\| |\psi_p|^2 \right\|_{L^{q/2}(\mathbb{R}^2)} \right)^{1/2} = \|\psi\|_{\widetilde{L^q}(\mathbb{R}^2,\mathcal{H})}. \tag{A.3}$$

This inequality becomes an equality in the special case $q = 2$ and we have the identification $\widetilde{L^2}(\mathbb{R}^2,\mathcal{H}) = L^2(\mathbb{R}^2,\mathcal{H})$.

This functional space $\widetilde{L^q}(\mathbb{R}^2,\mathcal{H})$ can be identified with the space $L^q(\lambda)$ introduced in [5, Definition 2.1] (in dimension 2 instead of dimension 3), with the choice $\lambda = (1,1,1,\cdots)$ and if ψ is identified with the sequence of its components $(\psi_p)_{p\in\mathbb{N}^*}$.

Each component ψ_p satisfies the equation

$$\left\{ \begin{array}{l} i\partial_t \psi_p = -\dfrac{1}{2}\Delta_x \psi_p + g_p \\ \psi_p(0,\cdot) = \psi_{0,p} \end{array} \right.$$

where $g_p = (g, \chi_p)_{\mathcal{H}}$ and $\psi_{0,p} = (\psi_0, \chi_p)_{\mathcal{H}}$. Therefore, an application of [5, Theorem 2.1] (adapted to dimension 2) gives:

$$\begin{aligned}\|\psi\|_{L^{q^*}((0,T),\widetilde{L^q}(\mathbb{R}^2,\mathcal{H}))} &\leq C\|\psi_0\|_{\widetilde{L^2}(\mathbb{R}^2,\mathcal{H})} + C\|g\|_{L^1((0,T),\widetilde{L^2}(\mathbb{R}^2,\mathcal{H}))} \\ &= C\|\psi_0\|_{L^2(\mathbb{R}^2,\mathcal{H})} + C\|g\|_{L^1((0,T),L^2(\mathbb{R}^2,\mathcal{H}))}.\end{aligned}$$

We conclude the proof by using (A.3). $\square$

Let now $\mathbb{A}$ be an unbounded operator on $\mathcal{H} = L^2(\mathbb{R})$ with the domain $\mathcal{D}(\mathbb{A})$. We assume that the operator $\mathbb{A}$ is self-adjoint and denote by $e^{it\mathbb{A}}$ the unitary group generated by $i\mathbb{A}$ on $\mathcal{H}$. In the paper, the results of the appendix are applied to the operator $\mathbb{A} = -\frac{1}{2}\frac{d^2}{dz^2} + V_c^\varepsilon$. The operator $i(\frac{1}{2}\Delta_x - \mathbb{A})$, defined, with an abuse of notation, as $i(\frac{1}{2}\Delta_x \otimes \mathbb{I}_{\mathcal{H}} - \mathbb{I}_{L^2(\mathbb{R}^2)} \otimes \mathbb{A})$ on $H^2(\mathbb{R}^2, \mathcal{H}) \cap L^2(\mathbb{R}^2, \mathcal{D}(\mathbb{A}))$, generates a group of isometries on $L^2(\mathbb{R}^2, \mathcal{H}) = L^2(\mathbb{R}^3)$. Let us now consider the problem

$$\begin{cases} i\partial_t \psi = -\dfrac{1}{2}\Delta_x \psi + \mathbb{A}\psi + f \\ \psi(0, x, z) = \psi_0. \end{cases} \tag{A.4}$$

where the source term $f(t, x, z)$ is given. The following result holds

Lemma A.2 *Let $\psi_0 \in L^2(\mathbb{R}^3)$ and $f \in L^1((0,T), L^2(\mathbb{R}^3))$. Then for any $q \in [2, \infty)$, the solution ψ of the Schrödinger equation* (A.4) *belongs to $L_t^{q^*} L_x^q L_z^2((0,T) \times \mathbb{R}^3)$ and satisfies*

$$\|\psi\|_{q^*,q,2} \leq C\|\psi_0\|_{L^2(\mathbb{R}^3)} + C\|f\|_{L^1((0,T),L^2(\mathbb{R}^3))}, \tag{A.5}$$

where C denotes a constant independent of the operator $\mathbb{A}$.

Proof. This lemma is a consequence of the above Lemma A.1. Let us denote $\phi(t, x, z) = e^{i\mathbb{A}t}\,\psi(t, x; z)$. Since $\mathbb{A}$ commutes with ∂_t and Δ_x, we have clearly

$$\begin{cases} i\partial_t \phi = -\Delta_x \phi + e^{i\mathbb{A}t}\, f \\ \phi(0, x, z) = \psi_0. \end{cases}$$

Therefore ϕ satisfies (A.1) with $g = e^{i\mathbb{A}t}\, f$. We conclude the proof by using (A.2), since $e^{i\mathbb{A}t}$ is an isometry on $L^2(\mathbb{R})$. $\square$

B The Poisson equation with $L_x^p L_z^1$ densities

This section deals with the convolution product

$$u = \frac{1}{r} * f,$$

where $r = \sqrt{|x|^2 + z^2}$ and $f \in L_x^p L_z^1$. We recall that throughout this paper $x \in \mathbb{R}^2$, $z \in \mathbb{R}$ and $L_x^p L_z^q = L^p(\mathbb{R}^2, L^q(\mathbb{R}))$. We first prove the following result in $\mathbb{R}^2$ with a convolution kernel more singular than the kernel of the Poisson equation:

Lemma B.1 *(i) Let $f \in L^p(\mathbb{R}^2)$ with $1 < p < 2$. Then*

$$\left\| \frac{1}{|x|} *_x f \right\|_{L^{p^\#}(\mathbb{R}^2)} \leq C_p \, \|f\|_{L^p(\mathbb{R}^2)}, \tag{B.1}$$

where $p^\# = \frac{2p}{2-p}$.
(ii) Let $f \in L^p(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $2 < p \leq +\infty$. Then

$$\left\| \frac{1}{|x|} *_x f \right\|_{L^\infty(\mathbb{R}^2)} \leq C_p \, \|f\|^\theta_{L^p(\mathbb{R}^2)} \|f\|^{1-\theta}_{L^1(\mathbb{R}^2)}, \tag{B.2}$$

where $\theta = \frac{p}{2p-2}$.

Proof. The first part of the lemma is a straightforward consequence of generalized Young's formula [18]. Indeed, the function $x \mapsto \frac{1}{|x|}$ belongs to $L^2_w(\mathbb{R}^2)$ and the function f is in $L^p(\mathbb{R}^2)$, thus $\frac{1}{|x|} *_x f$ belongs to $L^{p^\#}(\mathbb{R}^2)$, with $\frac{1}{p} + \frac{1}{2} = 1 + \frac{1}{p^\#}$.

In order to prove Item *(ii)*, for any $R > 0$ we cut the integral into two parts:

$$\begin{aligned} \left| \frac{1}{|x|} *_x f \right| &\leq \int_{|x-x'|<R} \frac{|f(x')|}{|x-x'|} dx' + \frac{1}{R} \|f\|_{L^1(\mathbb{R}^2)} \\ &\leq C R^{\frac{p-2}{p}} \|f\|_{L^p(\mathbb{R}^2)} + \frac{1}{R} \|f\|_{L^1(\mathbb{R}^2)}, \end{aligned}$$

where we used the Hölder's inequality to estimate the first integral. The value of θ is obtained after an optimisation of R. □

Lemma B.2 *(i) Let $f \in L^p_x L^1_z$ with $1 < p < 2$. Then we have*

$$\left\| \frac{1}{r} * f \right\|_{p^\#,\infty} + \left\| \nabla_{x,z} \left(\frac{1}{r} * f \right) \right\|_{p^\#,1} \leq C_p \, \|f\|_{p,1}, \tag{B.3}$$

where $p^\# = \frac{2p}{2-p}$. If in addition $\nabla_x f \in L^p_x L^1_z$ then

$$\left\| \nabla_{x,z} \left(\frac{1}{r} * f \right) \right\|_{p^\#,\infty} \leq C_p \, \|\nabla_x f\|_{p,1}. \tag{B.4}$$

(ii) Let $f \in L^p_x L^1_z \cap L^1(\mathbb{R}^3)$ with $2 < p \leq +\infty$. Then we have

$$\left\| \frac{1}{r} * f \right\|_{L^\infty(\mathbb{R}^3)} + \left\| \nabla_{x,z} \left(\frac{1}{r} * f \right) \right\|_{\infty,1} \leq C_p \, \|f\|^\theta_{p,1} \, \|f\|^{1-\theta}_{L^1(\mathbb{R}^3)}, \tag{B.5}$$

where $\theta = \frac{p}{2p-2}$. If in addition $\nabla_x f \in L^p_x L^1_z \cap L^1(\mathbb{R}^3)$ then

$$\left\| \nabla_{x,z} \left(\frac{1}{r} * f \right) \right\|_{L^\infty(\mathbb{R}^3)} \leq C_p \, \|\nabla_x f\|^\theta_{p,1} \, \|\nabla_x f\|^{1-\theta}_{L^1(\mathbb{R}^3)}. \tag{B.6}$$

Proof. The two Items *(i)* and *(ii)* can be proved similarly by using respectively Items *(i)* and *(ii)* of Lemma B.1. We shall only prove here Item *(i)*. Denoting $u = \frac{1}{r} * f$, we have

$$\|u(x,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{|x|} *_x \|f(x,\cdot)\|_{L^1(\mathbb{R})}$$

and the first part of (B.3) is a consequence of (B.1), since $x \mapsto \|f(x,\cdot)\|_{L^1(\mathbb{R})}$ belongs to $L^p(\mathbb{R}^2)$. Now we have

$$\begin{aligned}
\int_{\mathbb{R}} |\nabla_x u(x,z)|\, dz \ &\le \iiint_{\mathbb{R}^4} \frac{|x-x'|}{(|x-x'|^2+(z-z')^2)^{3/2}} |f(x',z')|\, dx'\, dz'\, dz. \\
&= 2\int_{\mathbb{R}^2} \frac{1}{|x-x'|} \|f(x',\cdot)\|_{L^1(\mathbb{R})}\, dx' \\
&= \frac{2}{|x|} *_x \|f(x,\cdot)\|_{L^1(\mathbb{R})},
\end{aligned}$$

where we have just evaluated the integral

$$\int_{\mathbb{R}} \frac{|x-x'|}{(|x-x'|^2+(z-z')^2)^{3/2}}\, dz = \frac{2}{|x-x'|}.$$

Then by using again (B.1) we conclude to the estimate of $\|\nabla_x u\|_{p^\#,1}$. We estimate $\|\partial_z u\|_{p^\#,1}$ similarly:

$$\begin{aligned}
\int_{\mathbb{R}} |\partial_z u(x,z)|\, dz \ &\le \iiint_{\mathbb{R}^4} \frac{|z-z'|}{(|x-x'|^2+(z-z')^2)^{3/2}} |f(x',z')|\, dx'\, dz'\, dz. \\
&= 2\int_{\mathbb{R}^2} \frac{1}{|x-x'|} \|f(x',\cdot)\|_{L^1(\mathbb{R})}\, dx'.
\end{aligned}$$

This proves (B.3). Next, in order to prove (B.4) and for $i = 1,2$ we write

$$\|\nabla_{x,z}\, \partial_{x_i} u\|_{p^\#,1} \le \left\| \nabla_{x,z} \left(\frac{1}{r} * (\partial_{x_i} f) \right) \right\|_{p^\#,1} \le C_p \,\|\partial_{x_i} f\|_{p,1}, \tag{B.7}$$

Together with (B.3) this implies that $\partial_{x_i} u$ belongs to $L^{p^\#}(\mathbb{R}^2, W^{1,1}(\mathbb{R}))$. Remark that $W^{1,1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Therefore, $\partial_{x_i} u$ is in $L^{p^\#}_x L^\infty_z$ and satisfies

$$\|\partial_{x_i} u\|_{p^\#,\infty} \le C \|\partial_z \partial_{x_i} u\|_{p^\#,1} \le C_p \,\|\partial_{x_i} f\|_{p,1}.$$

To prove (B.4), it remains to estimate $\partial_z u$. We recall that $-\Delta_{x,z} u = f$. Besides, we remark that

$$x \mapsto \|f(x,\cdot)\|_{L^1(\mathbb{R})}$$

belongs to $W^{1,p}(\mathbb{R}^2) \hookrightarrow L^{p^\#}(\mathbb{R}^2)$. Consequently,

$$\|f\|_{p^\#,1} \le C_p \|\nabla_x f\|_{p,1}$$

and applying (B.7), we get

$$\|\partial_{zz}u\|_{p^\#,1} \leq \|f\|_{p^\#,1} + \|\Delta_x u\|_{p^\#,1} \leq C_p \, \|\nabla_x f\|_{p,1}.$$

Therefore, as above, $\partial_z u$ is bounded in $L^{p^\#}(\mathbb{R}^2, W^{1,1}(\mathbb{R}))$, thus in $L_x^{p^\#} L_z^\infty$. $\square$

Lemma B.3 *(i) Let $f \in L_x^p L_z^1$, with $1 < p < \infty$, be such that $\int_{\mathbb{R}} f(x,z)dz = 0$, x a.e. and $z\, f \in L_x^p L_z^1$. Then for any $\alpha \in (0, \min(1, 2/p))$ we have*

$$\left\| \frac{1}{r} * f \right\|_{q,\infty} \leq C \, \|zf\|_{p,1}^{1-\alpha} \|f\|_{p,1}^{\alpha}, \tag{B.8}$$

where $q = \frac{2p}{2-\alpha p}$.
(ii) Let $f \in L_x^p L_z^1$, with $2 < p < \infty$ be such that $\int_{\mathbb{R}} f(x,z)dz = 0$, x a.e. and $z\, f \in L_x^p L_z^1$. Then

$$\left\| \frac{1}{r} * f \right\|_{L^\infty(\mathbb{R}^3)} \leq C \, \|zf\|_{p,1}^{1-2/p} \|f\|_{p,1}^{2/p} \tag{B.9}$$

Proof. Denote $u = \frac{1}{r} * f$. Since $\int_{\mathbb{R}} f(x,z)dz = 0$, we have

$$\begin{aligned} u(x,z) &= \iint_{\mathbb{R}^3} \left(\frac{1}{(|x-x'|^2 + (z-z')^2)^{1/2}} - \frac{1}{(|x-x'|^2 + z^2)^{1/2}} \right) f(x',z')\, dx'dz' \\ &= \iint_{\mathbb{R}^3} \int_0^{z'} \left(\frac{z-\xi}{(|x-x'|^2 + (z-\xi)^2)^{3/2}} \right) f(x',z')\, d\xi dx'dz'. \end{aligned} \tag{B.10}$$

Then we remark that for any z, ξ and $x \neq x'$, we have

$$\frac{|z-\xi|}{(|x-x'|^2 + (z-\xi)^2)^{3/2}} \leq \frac{2}{3\sqrt{3}} \frac{1}{|x-x'|^2} \tag{B.11}$$

and that

$$\int_0^{z'} \frac{|z-\xi|}{(|x-x'|^2 + (z-\xi)^2)^{3/2}} d\xi \leq \int_{\mathbb{R}} \frac{|z-\xi|}{(|x-x'|^2 + (z-\xi)^2)^{3/2}} d\xi = \frac{2}{|x-x'|}. \tag{B.12}$$

Let us first prove (B.8). By (B.11) and (B.12) we have for any $0 \leq \alpha \leq 1$

$$\begin{aligned} \int_{\mathbb{R}} \int_0^{z'} \frac{|z-\xi|}{(|x-x'|^2 + (z-\xi)^2)^{3/2}} |f(x',z')| d\xi dz' &\leq C \int_{\mathbb{R}} \frac{|z' f(x',z')|^{1-\alpha}}{|x-x'|^{2(1-\alpha)}} \frac{|f(x',z')|^{\alpha}}{|x-x'|^{\alpha}} dz' \\ &\leq \frac{C}{|x-x'|^{2-\alpha}}\, g(x). \end{aligned}$$

where

$$g(x) = \left(\int |zf(x,z)|dz \right)^{1-\alpha} \left(\int |f(x,z)|dz \right)^{\alpha}.$$

Hence from (B.10) we deduce that

$$\|u(x,\cdot)\|_{L^\infty(\mathbb{R})} \leq C\left(\frac{1}{|x|^{2-\alpha}} * g\right)(x).$$

From the assumptions on f, we deduce that g belongs to $L^p(\mathbb{R}^2)$. Since the function $x \mapsto \frac{1}{|x|^{2-\alpha}}$ belongs to $L_w^{2/(2-\alpha)}(\mathbb{R}^2)$, the generalized Young's inequality gives (B.8).

In order to prove (B.9), the right-hand side of (B.10) is cut into two parts

$$\int_{\mathbb{R}^3} = \int_{|x-x'|>R} + \int_{|x-x'|<R}.$$

By (B.11), the first part is controlled by

$$C\int_{|x-x'|>R} \frac{\|z'f(x',\cdot)\|_{L^1(\mathbb{R})}}{|x-x'|^2}dx' \leq \frac{C}{R^{2/p}}\|z'f\|_{p,1},$$

while the second integral is estimated, through (B.12), by

$$C\int_{|x-x'|<R} \frac{\|f(x',\cdot)\|_{L^1(\mathbb{R})}}{|x-x'|}dx' \leq R^{1-2/p}\|f\|_{p,1}.$$

An optimisation of R leads to (B.9). $\square$

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