On a Vlasov-Schrödinger-Poisson model

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Abstract

Weak solutions of a Vlasov-Schrödinger-Poisson system are shown to exist in the stationary and time-dependent situations. This system models the transport and interaction of electrons in a bidimensional electron gas. The particles are assumed to have a wave behaviour in the confinement directions (z) and to behave like point particles in the directions parallel to the electron gas (x). For each fixed x and at each time t, the eigenfunctions and the eigen-energies of the Schrödinger operator in the z are computed. The occupation number of each eigenfunction is computed through the resolution of a Vlasov equation in the x direction, the force field being the gradient of the eigen-energy. The whole system is coupled to the Poisson equation for the electrostatic interaction. Existence of weak solutions is shown for boundary value problems in the stationary and time-dependent regimes. The proofs rely on the one hand on the study of quasistatic Schrödinger-Poisson systems and on the other hand on an energy estimate (for the time-dependent case) and on supersolution techniques (for the stationary case).

1 Introduction

Classical motion of charged particles (say electrons) can be described by kinetic equations (Vlasov, Boltzmann) coupled to Poisson equation for the electrostatic forces [30, 12, 33, 4, 1, 37, and references therein]. For ultrasmall electron systems, like nanostructures, quantum effects like tunneling are important [46, 18, 25]. The system is then well described by the Schrödinger-Poisson system [15, 34, 31, 40, 41, 42, 11]. In various situations, like in resonant tunneling diodes [17, 24, 39], quantum effects occur in some parts of the electron ensemble while other parts exhibit purely classical behaviour. A sound description of such systems requires the use of the Schrödinger equation when necessary and kinetic (or fluid) equations otherwise. This leads to spatial coupling strategies between quantum and classical models which is the subject of an intensive research effort [5, 6, 19].

In partially confined electron systems like two dimensional electron gases (2DEG), nanotubes or nanowires, the quantum-classical coupling has different features. Indeed, the width of a two-dimensional electron gas lying at a heterojunction (like

Silicon-Oxide junctions in field effect transistors, or GaAs-AlGaAs junctions in modulation doped structures) is a few nanometers. As this length is comparable to the electron de Broglie length, the description of transport phenomena necessitates the use of the Schrödinger equation. In the direction parallel to the heterojunction, the lengthscale is usually several times higher, and a classical description for electron transport is suitable. This leads to a coupling between classical and quantum models in momentum space. The so-called subband models [2, 3, 18, 25] which rely on the Born-Oppenheimer approximation allow such coupling. They have been recently derived by the authors in [8] thanks to a partial semi-classical limit of the Schrödinger equation (see [29, 45] and references therein for a related approach in molecular dynamics). The aim of this paper is to analyze a ballistic subband model coupled to the Poisson equation. All along the paper, the confined direction is denoted by $z \in (0,1)$ (the study is restricted to one-dimensional confinement), while the non-confined direction is called $x \in \omega$, where ω is a bounded regular domain of $\mathbb{R}^d$.

In dimensionless variables, the problem consists in finding, for $t \in (0,T)$, $x \in \omega$, $z \in (0,1)$ and $v \in \mathbb{R}^d$, the unknowns $V(t,x,z)$, $(\epsilon_p(t,x), \chi_p(t,x,z), f_p(t,x,v))_{p\in\mathbb{N}^*}$ solving

$$\partial_t f_p + v \cdot \nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = 0, \tag{1.1}$$

$$\begin{cases} -\dfrac{1}{2}\,\partial_{zz}\chi_p + (V + V_{ext})\chi_p = \epsilon_p \chi_p\,, \\ \chi_p(t,x,\cdot) \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,, \end{cases} \tag{1.2}$$

$$-\Delta V = \sum_{p\geq 1} \left(\int_{\mathbb{R}^d} f_p\,dv \right) |\chi_p|^2\,, \tag{1.3}$$

with suitable initial and boundary conditions that will be specified later on. In this system the symbol Δ is the Laplace operator in the (x,z) variables. The function V is the selfconsistent electrostatic potential while V_{ext} is a given external potential. The functions ϵ_p, χ_p, f_p are the energy, the wave function and the distribution function of the p-th subband. For a given potential V, the problem (1.2) is an eigenvalue problem in the z variable, parametrized by t and x. The study is restricted to one-dimensional confinement in order to avoid energy crossing and to ensure the regularity, with respect to V, of the wave functions and energies of subbands.

The problem is quantum and quasistatic in the z direction while it is classical and evolutionary in the x direction. A special solution of (1.1)–(1.3) can be constructed by assuming invariance with respect to x, which leads to the stationary one-dimensional Schrödinger-Poisson system

$$\begin{cases} -\dfrac{1}{2}\,\dfrac{d^2\chi_p}{dz^2} + (V + V_{ext})\chi_p = \epsilon_p \chi_p\,, \\ -\dfrac{d^2 V}{dz^2} = \displaystyle\sum_{p\geq 1} \lambda_p\, |\chi_p|^2\,. \end{cases}$$

A more general version of this Schrödinger-Poisson problem is treated by Nier [40] by a variational technique (se also [41, 42] for the same problem in higher dimensions). We shall employ some of the results and techniques of [40], to analyze the "Schrödinger part" of our system. Concerning the "Vlasov aspect", we shall make use of the standard L^p, interpolation and energy estimates satisfied by weak and renormalized solutions [30, 12, 33, 1, 20, 21, 36, 37, 38].

2 Definitions and main results

The variable x lies in a bounded regular domain $\omega \subset \mathbb{R}^d$, where $d = 1$, $d = 2$ or $d = 3$. We set $\Omega = \omega \times (0, 1)$ and denote by $\nu(x)$ the outward unit normal vector at $x \in \partial\omega$. The domain Ω represents the spatial domain where transport and interaction will be studied. The case $d = 3$ is included in our analysis, although it corresponds to a four dimensional position space ! The incoming/outgoing sets are defined by

$$\Sigma_\pm = \{(x, v) \in \Sigma;\ \pm v\cdot\nu(x) > 0\}, \qquad \text{where } \Sigma = \partial\omega \times \mathbb{R}^d, \tag{2.1}$$

and equipped with the measure

$$d\Sigma(x, v) = |v \cdot \nu(x)|\,d\sigma(x)\,dv, \tag{2.2}$$

where $d\sigma$ is the surface measure on $\partial\omega$. The $p-$th subband surfacic charge density and surfacic current density are defined by

$$\rho_p = \int_{\mathbb{R}^d} f_p\,dv \quad ; \quad j_p = \int_{\mathbb{R}^d} v\, f_p\,dv$$

while the total (volume) charge density will be denoted by

$$n = \sum_{p\geq 1} \left(\int_{\mathbb{R}^d} f_p\,dv\right) |\chi_p|^2 = \sum_{p\geq 1} \rho_p\,|\chi_p|^2.$$

Definition 2.1 *(i) Let $1 \leq p, q \leq +\infty$. Then*

$$L^{p,q}(\Omega) = \left\{ u \in L^1_{loc}(\Omega), \quad \|u\|_{L^{p,q}(\Omega)} = \left(\int_\omega \|u(x,\cdot)\|^p_{L^q(0,1)}\,dx\right)^{\frac{1}{p}} < +\infty \right\} \tag{2.3}$$

(with an obvious generalization of this definition for $p = +\infty$).
(ii) Let $1 \leq r \leq +\infty$. Then

$$W^{1,r}_\omega = \left\{u \in W^{1,r}(\Omega), \quad u = 0 \ \text{on } \partial\omega \times (0,1)\right\}; \quad H^1_\omega = W^{1,2}_\omega. \tag{2.4}$$

(iii) If $1 \leq r < +\infty$ then $W^{-1,r'}_\omega$ is the dual of $W^{1,r}_\omega$, where $r' = \frac{r}{r-1}$.

For any Banach space E, we shall say that a sequence $(\rho_p)_{p\in\mathbb{N}^*}$ belongs to $\ell^q(E)$ if for all $p\geq 1$ we have $\rho_p \in E$ and if $(\sum_{p\geq 1}\|\rho_p\|^q)^{1/q} < +\infty$, the last quantity being the norm of $(\rho_p)_{p\in\mathbb{N}^*}$ in $\ell^q(E)$. The collection of $(\rho_p)_{p\in\mathbb{N}^*}$ will also be denoted by (ρ).

2.1 The stationary problem

Let us consider the stationary version of the Vlasov-Schrödinger-Poisson problem and make precise the boundary and initial conditions:

$$\begin{cases} v\cdot\nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = 0, \\ \gamma^- f_p = \alpha g_p + (1-\alpha)\mathcal{R}_p(\gamma^+ f_p) \quad \text{on } \Sigma_-, \end{cases} \tag{2.5}$$

$$\begin{cases} -\dfrac{1}{2}\,\partial_{zz}\chi_p + (V+V_{ext})\chi_p = \epsilon_p \chi_p\,, \\ \chi_p(x,\cdot)\in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,, \end{cases} \tag{2.6}$$

$$\begin{cases} -\Delta V = \displaystyle\sum_{p\geq 1}\left(\int_{\mathbb{R}^d} f_p\,dv\right)|\chi_p|^2\,, \\ V = 0 \quad \text{on } \partial\omega\times(0,1), \\ \partial_z V = 0 \quad \text{on } \omega\times\{0\}\,\cup\,\omega\times\{1\}. \end{cases} \tag{2.7}$$

The operators γ^- and γ^+ denote the trace operators respectively on Σ_- and Σ_+. For any $p\geq 1$ the boundary distribution function g_p is given and $\mathcal{R}_p$ is the elastic-diffusive boundary operator defined as in [7, 35] by

$$\mathcal{R}_p(f)(x,v) = \int_{v'\cdot\nu(x)>0} \sigma_p(x,v',v)\delta(|v|^2-|v'|^2)f(x,v')\,|v'\cdot\nu(x)|\,dv', \quad \forall (x,v)\in\Sigma_-. \tag{2.8}$$

We shall make the following assumptions on the data of the stationary problem:

Assumptions:

(**HS-1**) $0<\alpha\leq 1$ and σ_p is a nonnegative function such that

$$\int_{v\cdot\nu(x)<0} \sigma_p(x,v',v)\,\delta(|v|^2-|v'|^2)\,|v\cdot\nu(x)|\,dv = 1 \qquad \text{for } (x,v')\in\Sigma_+ \quad \text{(mass conservation)},$$

$$\sigma(x,v',v) = \sigma(x,-v,-v') \qquad \text{(reciprocity)}.$$

(**HS-2**) The external potential V_{ext} is nonnegative and lies in $C^2(\overline{\Omega})$.

(**HS-3**) There exists a sequence of nonincreasing functions G_p such that

$$0\leq g_p(x,v)\leq G_p\left(\frac{v^2}{2}+\frac{\pi^2p^2}{2}+\max_{\partial\omega\times[0,1]}(V_{ext})\right),$$

$$\text{with} \quad \sum_{p\geq 1} p \int_0^{+\infty} |u|^{d-1} G_p \left(\frac{u^2}{2} + \frac{\pi^2 p^2}{2} \right) du \ < +\infty.$$

Theorem 2.2 *Under Hypotheses* (**HS-1**)–(**HS-3**)*, the system* (2.5)–(2.7) *admits a weak solution* $(V, (\epsilon_p, \chi_p, f_p)_{p\in\mathbb{N}^*})$ *such that* $V \in W^{2,s}(\Omega)$*,* $\epsilon_p \in W^{2,s}(\omega)$*,* $\chi_p \in W^{2,s}(\Omega)$ *for all* $s < +\infty$ *and the total charge density* n *belongs to* $L^\infty(\Omega)$*. Moreover this solution satisfies the following pointwise estimate:*

$$0 \leq f_p(x,v) \leq G_p \left(\frac{v^2}{2} + \epsilon_p(x) \right), \quad (x,v) \in \omega \times \mathbb{R}^d.$$

2.2 The time-dependent problem

Now we turn to the time-dependent Vlasov-Schrödinger-Poisson problem and make precise the initial and boundary conditions:

$$\begin{cases} \partial_t f_p + v \cdot \nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = 0, \\ \gamma^- f_p = \alpha g_p + (1-\alpha) \mathcal{R}_p(\gamma^+ f_p) \quad \text{on } (0,T) \times \Sigma_- \,, \\ f_p(0,x,v) = f_{p,0}(x,v), \end{cases} \tag{2.9}$$

$$\begin{cases} -\dfrac{1}{2}\, \partial_{zz} \chi_p + (V + V_{ext}) \chi_p = \epsilon_p \chi_p \,, \\ \chi_p(t,x,\cdot) \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p \, \chi_q \, dz = \delta_{pq} \,, \end{cases} \tag{2.10}$$

$$\begin{cases} -\Delta V = \displaystyle\sum_{p\geq 1} \left(\int_{\mathbb{R}^d} f_p \, dv \right) |\chi_p|^2 \,, \\ V = 0 \quad \text{on } \partial\omega \times (0,1), \\ \partial_z V = 0 \quad \text{on } \omega \times \{0\} \ \cup \ \omega \times \{1\}. \end{cases} \tag{2.11}$$

The boundary operator $\mathcal{R}_p$ is the same as the one introduced in the stationary case with a possibly time-dependent cross section

$$\mathcal{R}_p(f)(t,x,v) = \int_{v'\cdot\nu(x)>0} \sigma_p(t,x,v',v) \, \delta(|v|^2 - |v'|^2) f(t,x,v') \, |v' \cdot \nu(x)| \, dv'$$

The following assumptions on the data of the time-dependent problem are made:

Assumptions:

(**H-1**) $\quad 0 \leq \alpha \leq 1$ and σ_p is a nonnegative function such that

$$\int_{v\cdot\nu(x)<0} \sigma_p(t,x,v',v) \delta(|v|^2 - |v'|^2) \, |v \cdot \nu(x)| \, dv = 1 \qquad \text{for } (t,x,v') \in (0,T) \times \Sigma_+$$

(mass conservation),

$$\sigma(t,x,v',v)=\sigma(t,x,-v,-v') \qquad \text{(reciprocity)}.$$

(**H-2**) The external potential V_{ext} lies in $C([0,T],C^2(\overline{\Omega}))\cap C^1([0,T],C(\overline{\Omega}))$.

(**H-3**) $\quad f_{p,0}\geq 0, \qquad ((1+v^2+p^2)f_{p,0})\in \ell^1(L^1(\omega\times\mathbb{R}^d))$.

(**H-4**) $\quad (f_{p,0})\in \ell^1(L^\infty(\omega\times\mathbb{R}^d))$.

(**H-5**) $\quad g_p\geq 0, \qquad ((1+v^2+p^2)g_p)\in \ell^1(L^1((0,T),L^1(\Sigma_-,d\Sigma)))$.

(**H-6**) $\quad (g_p)\in \ell^1(L^\infty((0,T)\times\Sigma_-))$.

Using the notations

$$M_0=\sum_{p\geq 1}\|f_{p,0}\|_{L^\infty(\Omega)} \quad ; \quad E_0=\sum_{p\geq 1}\iint (1+v^2+p^2)f_{p,0}\,dx\,dv \tag{2.12}$$

$$M_b=\sum_{p\geq 1}\|g_p\|_{L^\infty((0,T)\times\Sigma_-)} \quad ; \quad E_b=\sum_{p\geq 1}\int_0^T\iint_{\Sigma_-}(1+v^2+p^2)g_p(t,x,v)\,d\Sigma(x,v)dt, \tag{2.13}$$

the main theorem for the time-dependent problem is

Theorem 2.3 *Let $T>0$. Under the hypotheses* (**H-1**)–(**H-6**)*, there exists a bound $\overline{\mathcal{E}}$ depending on $\|V_{ext}(0)\|_{C^2}$ and on $\int_0^T\|[\partial_t V_{ext}]^+\|_{L^\infty(\Omega)}\,dt$ such that if*

$$(M_0+M_b)^{2/d}(E_0+E_b)<\overline{\mathcal{E}} \tag{2.14}$$

the system (2.9)-(2.10)-(2.11) *admits a weak solution* $(V,(\epsilon_p,\chi_p,f_p)_{p\in\mathbb{N}^*})$ *on* $[0,T]$ *in the following functional spaces*

$$V\in C([0,T],W^{2,\frac{d+2}{d}}(\Omega)), \qquad n\in C([0,T],L^{\frac{d+2}{d},\infty}(\Omega)),$$

$$\epsilon_p\in C([0,T],W^{2,\frac{d+2}{d}}(\omega)), \qquad \chi_p\in C([0,T],W^{2,\frac{d+2}{d}}(\Omega)).$$

2.3 Outline

An important part of the work relies on the analysis of the quasistatic Schrödinger-Poisson problem similar to the one studied by Nier in [40]. This is the object of Section 3. Section 4 is devoted to the proof of existence of stationary solutions for the Vlasov-Schrödinger-Poisson problem (2.5)–(2.7). It uses the supersolution technique developed by Poupaud [43]. We briefly describe the proof which relies on the Schauder fixed point theorem and stress on the differences between our problem and the standard Vlasov-Poisson problem. In section 5, the time-dependent problem is tackled with a fixed point procedure (different from the one used in the stationary case) which takes advantage of an energy estimate. This *a priori* estimate is obtained in Subsection 5.1. The properties of the Vlasov equation (2.9) are recalled in Subsection 5.2, while Subsection 5.3 is devoted to the analysis of a regularized Vlasov-Schrödinger-Poisson problem whose unregularized limit, performed in Subsection 5.4, finishes the proof of Theorem 2.3. Section 6 is devoted to some comments while the Appendix contains some basic spectral properties of one-dimensional Schrödinger operators and some results on Sobolev embeddings in the $L^{p,q}$ spaces.

3 Analysis of the Schrödinger-Poisson system

This section is devoted to the study of the Schrödinger-Poisson system

$$\left\{\begin{array}{l} -\dfrac{1}{2}\,\partial_{zz}\chi_p(t,x;z) + (V+V_{ext})\chi_p(t,x;z) = \epsilon_p(t,x)\chi_p(t,x;z), \\ \chi_p(t,x;.) \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,, \end{array}\right. \tag{3.1}$$

$$\left\{\begin{array}{l} -\Delta V = \displaystyle\sum_{p\geq 1}\rho_p(t,x)\ |\chi_p(t,x,z)|^2\,, \\ V=0 \quad \text{on } \partial\omega\times(0,1), \\ \partial_z V = 0 \quad \text{on } \omega\times\{0\}\ \cup\ \omega\times\{1\}. \end{array}\right. \tag{3.2}$$

We recall that $\omega\subset\mathbb{R}^d$ and shall not make in this section any restriction on the dimension $d\in\mathbb{N}^*$. The occupation numbers $\rho_p(t,x)$ are assumed to be given functions. They satisfy integrability conditions which shall be specified later on. Although in practice $\rho_p(t,x)$ are nonnegative, no sign assumption assumption is made in this section. Besides, the time t appears only as a parameter and is skipped for notational simplicity.

Let us first introduce a regularized version of the Schrödinger-Poisson system (3.1)-(3.2). First define the linear regularization operator by

$$\begin{array}{rccl} R^\varepsilon: & L^1(\Omega) & \to & C^\infty(\overline{\Omega}) \\ & V & \mapsto & R^\varepsilon[V](x,z) = \left(\overline{V} *_x \xi_{\varepsilon,x} *_z \xi_{\varepsilon,z}\right)\big|_{\overline{\Omega}} \end{array} \tag{3.3}$$

where $\overline{V}$ is the extension of V by zero outside Ω and $\xi_{\varepsilon,x}$ and $\xi_{\varepsilon,z}$ are C^∞ nonnegative compactly supported even approximations of the unity respectively on $\mathbb{R}^d$ and $\mathbb{R}$. Standard convolution results imply the following lemma, that we state without proof:

Lemma 3.1 *(i) The operator R^ε is a bounded operator on $L^{p,q}(\Omega)$ for $1\leq p,q\leq+\infty$ and satifies*

$$\forall V\in L^{p,q}(\Omega),\quad \|R^\varepsilon[V]\|_{L^{p,q}}\leq\|V\|_{L^{p,q}},$$
$$\text{if}\quad 1\leq p,q<+\infty\quad\text{then}\quad \lim_{\varepsilon\to 0}\|R^\varepsilon[V]-V\|_{L^{p,q}}=0.$$

(ii) Assume that $V \in C^0((\overline{\omega}), L^q(0,1))$ *for some* $q < +\infty$. *Then* $\lim_{\varepsilon\to 0} \|R^\varepsilon[V] - V\|_{L^{\infty,q}(\omega'\times(0,1))} = 0$ *for any open* ω' *such that* $\overline{\omega'} \subset \omega$. *If moreover* $V(x,\cdot) = 0$ *on* $\partial\omega$, *then* $\lim_{\varepsilon\to 0} \|R^\varepsilon[V] - V\|_{L^{\infty,q}(\Omega)} = 0$.
(iii) The operator R^ε *is selfadjoint on* $L^2(\Omega)$.
(iv) Let $r \geq 1$ *be given and let* $V \in W^{1,r}_\omega$. *Then*

$$\nabla_x R^\varepsilon[V] = R^\varepsilon[\nabla_x V] \quad ; \quad \lim_{\varepsilon\to 0} \|\nabla_x R^\varepsilon[V] - \nabla_x V\|_{L^r(\Omega)} = 0.$$

The regularized Schrödinger-Poisson system reads

$$\begin{cases} -\dfrac{1}{2}\,\partial_{zz}\chi_p(x;z) + (R^\varepsilon[V] + V_{ext})\chi_p(x;z) = \epsilon_p(x)\chi_p(x;z), \\ \chi_p(x,\cdot) \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,, \end{cases} \tag{3.4}$$

$$\begin{cases} -\Delta V = R^\varepsilon\left[\displaystyle\sum_{p\geq 1} \rho_p(x)\ |\chi_p(x;z)|^2\right], \\ V = 0 \quad \text{on } \partial\omega \times (0,1), \\ \partial_z V = 0 \quad \text{on } \omega\times\{0\}\ \cup\ \omega\times\{1\}. \end{cases} \tag{3.5}$$

Remark 3.2 When $\varepsilon = 0$, the regularization operator is $R_0 = Id$ and the regularized problem (3.4)-(3.5) reduces to the unregularized system (3.1)-(3.2).

The main results of this section are the following two theorems:

Theorem 3.3 (Existence and estimate) *Let* $\varepsilon \in [0,1]$ *and* $(\rho_p(x))_{p\geq 1}$ *be a set of occupation factors in* $\ell^1(L^q)$ *for some* q *satisfying* $\max(1, d/2) < q < +\infty$.
(i) Assume that $V_{ext} \in L^{q',\infty}(\Omega) \cap L^{\infty,1}(\Omega)$ *where* q' *is the conjugate coefficient of* q. *Then* (3.4)-(3.5) *admits a solution* $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon)_{p\geq 1})$ *such that* $V^\varepsilon \in W^{2,q}(\Omega)$.
(ii) Assume that $V_{ext} \in L^{\infty,1}(\Omega)$. *Let* $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon)_{p\geq 1})$ *be a solution of* (3.4)-(3.5) *such that* $V^\varepsilon \in W^{1,1}_\omega$. *Then* $V^\varepsilon \in W^{2,q}(\Omega)$ *and*

$$\|V^\varepsilon\|_{W^{2,q}(\Omega)} \leq C_{\rho,V_{ext}}\,,$$

where the constant $C_{\rho,V_{ext}}$ *only depends on* $\|\rho\|_{\ell^1(L^q)}$ *and* $\|V_{ext}\|_{L^{\infty,1}}$.

Theorem 3.4 (Uniqueness and continuity for small data) *Let* $\varepsilon \in [0,1]$ *and* $\max(1, d/2) < q < +\infty$. *Assume that* $V_{ext} \in L^{q',\infty}(\Omega) \cap L^{\infty,1}(\Omega)$. *There exists a constant* $\mathcal{N}$ *independent of* ε *such that:*
(i) If $\|\rho\|_{\ell^1(L^q)} \leq \mathcal{N}$, *the solution* $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon)_{p\geq 1})$ *of* (3.4)-(3.5) *is unique and satisfies*

$$\|V^\varepsilon\|_{W^{2,q}(\Omega)} \leq C\,\|\rho\|_{\ell^1(L^q)}.$$

Moreover, as $\varepsilon \to 0$, *the solution* V^ε *of the regularized problem* (3.4)-(3.5) *converges to the solution* V *of the unregularized problem* (3.1)-(3.2) *in the* $W^{2,q}(\Omega)$ *topology*

and uniformly with respect to (ρ).
(ii) If $(\rho_p)_{p\geq 1}$ and $(\tilde{\rho}_p)_{p\geq 1}$ are two sets of occupation factors such that $\|\rho\|_{\ell^1(L^q)} \leq \mathcal{N}$ and $\|\tilde{\rho}\|_{\ell^1(L^q)} \leq \mathcal{N}$ then the corresponding solutions satisfy

$$\|V^\varepsilon - \tilde{V}^\varepsilon\|_{W^{2,q}(\Omega)} \leq C\,\|\rho - \tilde{\rho}\|_{\ell^1(L^q)}.$$

(iii) If $\|\rho\|_{\ell^1(L^q)} \leq \mathcal{N}$ and $(p^2\rho_p)_{p\geq 1} \in \ell^1(L^1(\omega))$ then the kinetic and potential energies defined by

$$\mathcal{E}^\varepsilon_{kin,z} = \sum_{p\geq 1} \iint \frac{1}{2}\,|\partial_z \chi^\varepsilon_p|^2\,\rho_p\,dx\,dz \quad ; \quad \mathcal{E}^\varepsilon_{pot} = \sum_{p\geq 1} \iint V_{ext}\,\rho_p\,dx\,dz + \iint \frac{1}{2}|\nabla_{x,z}V^\varepsilon|^2\,dx\,dz$$

satisfy the estimates

$$\mathcal{E}^\varepsilon_{kin,z} \leq C \sum_{p\geq 1} p^2\,\|\rho_p\|_{L^1} \quad ; \quad \mathcal{E}^\varepsilon_{pot} \leq C\,\|\rho\|_{\ell^1(L^1)}. \tag{3.6}$$

In this theorem, the constants $\mathcal{N}$ and C only depend on $\|V_{ext}\|_{L^{\infty,1}}$.

The proofs of these theorems are developed in the three following subsections.

3.1 Step 1 : Construction of a solution in $H^1(\Omega)$

In order to construct a solution in H^1_ω of the regularized Schrödinger-Poisson system, we proceed analogously to [40] and notice that this problem has a variational structure. Indeed $(V^\varepsilon, (\epsilon^\varepsilon_p, \chi^\varepsilon_p)_{p\geq 1})$ is a weak solution of (3.4)-(3.5) if and only if V^ε is a critical point in H^1_ω of the functional

$$J_{\rho,\varepsilon}(V) = J^0(V) + J^1_{\rho,\varepsilon}(V), \tag{3.7}$$

where

$$J^0(V) = \frac{1}{2}\int_\Omega |\nabla_{x,z}V|^2\,dx\,dz$$

and

$$J^1_{\rho,\varepsilon}(V) = \sum_{p\geq 1} \int_\omega \left(\epsilon_p[V_{ext}(x,\cdot)] - \epsilon_p[R^\varepsilon[V](x,\cdot) + V_{ext}(x,\cdot)]\right)\,\rho_p(x)\,dx\,.$$

The function $\epsilon_p[U]$ is the one defined in Appendix A. We shall prove that, under slightly different hypotheses from those of Theorem 3.3, the above functional has a minimizer and that this minimizer defines a solution of (3.1)-(3.2).

Lemma 3.5 *Let $\varepsilon \in [0,1]$. Assume that $(\rho) \in \ell^1(L^q)$ for some $q > \frac{2d}{d+1}$ and that $V_{ext} \in L^{q',\infty}(\Omega)$. The functional $J_{\rho,\varepsilon}$ defined in (3.7) is continuous, locally Lipschitz and weakly lower semicontinuous on H^1_ω. It is coercive : there exists a constant C_q depending on q (and not on ε) such that*

$$J_{\rho,\varepsilon}(V) \geq \frac{1}{2}\|\nabla_{x,z}V\|^2_{L^2} - C_q\|\rho\|_{\ell^1(L^q)}\|\nabla_{x,z}V\|_{L^2}.$$

Proof. We have $J_{\rho,\varepsilon} = J^0 + J^1_{\rho,\varepsilon}$. The first functional J^0 is continuous and weakly lower semicontinuous on $H^1(\Omega)$, while the second one $J^1_{\rho,\varepsilon}$ satisfies

$$|J^1_{\rho,\varepsilon}(U) - J^1_{\rho,\varepsilon}(V)| \leq C \sum_{p\geq 1} \|\epsilon_p[R^\varepsilon[U] + V_{ext}] - \epsilon_p[R^\varepsilon[V] + V_{ext}]\|_{L^{q'}} \|\rho_p\|_{L^q}$$

where $q' < \frac{2d}{d-1}$ is the conjugate coefficient of q. Lemmas A.1 and 3.1 imply

$$|J^1_{\rho,\varepsilon}(U) - J^1_{\rho,\varepsilon}(V)| \leq C\|\rho\|_{\ell^1(L^q)}\|R^\varepsilon[U-V]\|_{L^{q',\infty}(\Omega)} \leq C\|\rho\|_{\ell^1(L^q)}\|U-V\|_{L^{q',\infty}(\Omega)}.$$

Therefore $J^1_{\rho,\varepsilon}$ is Lipschitz on $L^{q',\infty}(\Omega)$ (with an ε-independent Lipschitz constant). Since $H^1(\Omega)$ is compactly embedded in $L^{q',\infty}(\Omega)$ (see Lemma B.1), $J^1_{\rho,\varepsilon}$ is Lipschitz and weakly continuous on $H^1(\Omega)$. Finally, the coercivity inequality on $J_{\rho,\varepsilon}$ can be deduced from $J^1_{\rho,\varepsilon}(0) = 0$ and Poincaré's inequality which is satisfied on H^1_ω. □

We are now able to prove the

Proposition 3.6 *Let $\varepsilon \in [0,1]$ and $(\rho_p(x))_{p\geq 1}$ be a set of occupation factors in $\ell^1(L^q)$ for some $q > \frac{2d}{d+1}$. Assume that $V_{ext} \in L^{q',\infty}(\Omega)$, where q' is the conjugate of q. Then the system (3.4)-(3.5) admits a solution $(V^\varepsilon, (\epsilon^\varepsilon_p, \chi^\varepsilon_p)_{p\geq 1})$ such that $V^\varepsilon \in H^1(\Omega)$.*

Proof. Lemma 3.5 yields the existence of a minimizer V^ε to $J_{\rho,\varepsilon}$. Since $J_{\rho,\varepsilon}(0) = 0$, the coercivity inequality implies that $\|\nabla_{x,z} V^\varepsilon\|_{L^2} \leq 2C_q\|\rho\|_{\ell^1(L^q)}$. The only thing left to show is that V^ε satisfies (3.4)-(3.5). This is a consequence of the following Lemma 3.7. □

Lemma 3.7 *Assume that $\rho \in \ell^1(L^q)$ for some $q > \frac{2d}{d+1}$ and that $V_{ext} \in L^{q',\infty}(\Omega)$, where q' is the conjugate of q. Then for all $U, V \in L^{q',\infty}(\Omega)$, we have*

$$\lim_{t\to 0} \frac{J^1_{\rho,\varepsilon}(V+tU) - J^1_{\rho,\varepsilon}(V)}{t} = \\ = -\int_\Omega R^\varepsilon\left[\sum_{p\geq 1} \rho_p(x)\, |\chi_p[(R^\varepsilon[V] + V_{ext})(x,\cdot)]|^2(z)\right] U(x,z)\, dx\, dz.$$

Proof. Let

$$\delta_t = \frac{J^1_{\rho,\varepsilon}(V+tU) - J^1_{\rho,\varepsilon}(V)}{t}.$$

Then we have

$$\delta_t = -\int_\omega \sum_{p\geq 1} \frac{\epsilon_p[(R^\varepsilon[V] + V_{ext} + tR^\varepsilon[U])(x,\cdot)] - \epsilon_p[(R^\varepsilon[V] + V_{ext})(x,\cdot)]}{t}\, \rho_p(x)\, dx.$$

Pointwise in x, the function $(t,z) \mapsto V(x,z) + tU(x,z)$ belongs to $L^\infty((0,1)^2)$. Hence Lemma A.6 applies and from (A.15) we deduce that each term of the integrand of δ_t converges almost everywhere towards

$$-\rho_p(x) \int_0^1 |\chi_p[(R^\varepsilon[V] + V_{ext})(x,\cdot)]|^2 (z) R^\varepsilon[U](x,z) dz.$$

The integration with respect to x and summation over p is done thanks to the Lebesgue dominated convergence theorem. Indeed, by (A.7) and Lemma 3.1, each term is bounded by $\|U(x,\cdot)\|_{L_z^\infty(0,1)} |\rho_p(x)|$, which belongs to $\ell^1(L^1(\omega))$. Therefore

$$\lim_{t \to 0} \delta_t = - \int_\omega \rho_p(x) \int_0^1 |\chi_p[(R^\varepsilon[V] + V_{ext})(x,\cdot)]|^2 (z) R^\varepsilon[U](x,z) dz$$

which finishes the proof since R^ε is symmetric on $L^2(\Omega)$. □

In the special case where the ρ_p are decreasing with respect to p, the solution of (3.4)-(3.5) can be shown to be unique. The following result is independent of Theorems 3.3 and 3.4 and is true even for large data ρ_p :

Proposition 3.8 *Let $\varepsilon \in [0,1]$ and $(\rho_p(x))_{p \geq 1}$ be a set of occupation factors in $\ell^1(L^q)$ for some $q > \frac{2d}{d+1}$. Assume that $V_{ext} \in L^{q',\infty}(\Omega)$, where q' is the conjugate of q. If $\rho_{p+1}(x) \leq \rho_p(x)$ for all $(p,x) \in \mathbb{N}^* \times \omega$ then the system (3.1)-(3.2) admits a unique solution such that $V \in H^1_\omega$.*

Proof. Only the uniqueness has to be proved. To this aim, we proceed analogously to [40] and prove that $J_{\rho,\varepsilon}$ is strictly convex. But since J^0 is itself strictly convex, it is enough to show that $J^1_{\rho,\varepsilon}$ is convex. We recall that, by Lemma 3.5, $J^1_{\rho,\varepsilon}$ is continuous on $H^1(\Omega)$. Similarly, one can show that for any fixed $V \in H^1(\Omega)$, the functional $V_{ext} \mapsto J^1_{\rho,\varepsilon}(V)$ is continuous on $L^{q',\infty}(\Omega)$. Therefore by a density argument it is enough to show that for $V_{ext} \in L^\infty(\Omega)$ the functional $J^1_{\rho,\varepsilon}$ is convex on $L^\infty(\Omega)$.

To this aim, we apply Lemma 3.7, then (A.17) to deduce that $J^1_{\rho,\varepsilon}$ is twice Gateaux differentiable on $L^\infty(\Omega)$ and satisfies

$$\begin{aligned} d^2 J^1_{\rho,\varepsilon}[V] W \cdot W &= -2 \sum_p \sum_{q \neq p} \int_\omega \frac{\rho_p}{\epsilon_p - \epsilon_q} \left(\int_0^1 \chi_p \chi_q \, R^\varepsilon[W] \, dz \right)^2 dx \\ &= \sum_p \sum_{q \neq p} \int_\omega \frac{\rho_p - \rho_q}{\epsilon_q - \epsilon_p} \left(\int_0^1 \chi_p \chi_q \, R^\varepsilon[W] \, dz \right)^2 dx \geq 0. \end{aligned}$$

This shows the convexity of $J^1_{\rho,\varepsilon}$ on $L^\infty(\Omega)$. □

3.2 Step 2 : Estimate in $W^{2,q}(\Omega)$ (proof of Theorem 3.3)

Proof of Item *(ii)***.** Let $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon)_{p\geq 1})$ be a solution of (3.4)-(3.5) such that $V^\varepsilon \in W^{1,1}_\omega$. Assume first that the ρ_p's are nonnegative. By the maximum principle, V^ε is nonnegative. Therefore, the function

$$W^\varepsilon(x) = \|V^\varepsilon(x,\cdot)\|_{L^1_z(0,1)} = \int_0^1 V^\varepsilon(x,z)\,dz$$

satisfies the equation

$$\begin{aligned} -\Delta_x W^\varepsilon(x) &= \int_0^1 R^\varepsilon\left[\sum_{p\geq 1}\rho_p(x)\ |\chi_p(x,\cdot)|^2\right]dz \\ &= \sum_{p\geq 1}\left[\rho_p(x)\ \|\chi_p(x,\cdot)\|^2_{L^2_z(0,1)}\right] * \xi_{\varepsilon,x} = \sum_{p\geq 1}\rho_p * \xi_{\varepsilon,x} \in L^q(\omega). \end{aligned}$$

Standard elliptic regularity results insure that $W^\varepsilon \in W^{2,q}(\omega)$ and that its norm depends only on $\|\rho\|_{\ell^1(L^q)}$. Besides, since $q > \max(1, d/2)$, $W^{2,q}(\omega)$ is embedded in $L^\infty(\omega)$. Hence $V^\varepsilon \in L^{\infty,1}(\Omega)$.

In the general case where ρ_p does not have a constant sign, denoting

$$\rho_p = \rho_p^+ - \rho_p^- \quad \text{and} \quad |\rho_p| = \rho_p^+ + \rho_p^-,$$

V^ε can be written $V^\varepsilon = V^{\varepsilon,+} - V^{\varepsilon,-}$, with

$$\begin{cases} -\Delta V^{\varepsilon,\pm} = R^\varepsilon\left[\sum_{p\geq 1}\rho_p^\pm\ |\chi_p[V^\varepsilon]|^2\right], \\ V^{\varepsilon,\pm} = 0 \quad \text{on } \partial\omega\times(0,1), \\ \partial_z V^{\varepsilon,\pm} = 0 \quad \text{on } \omega\times\{0\}\ \cup\ \omega\times\{1\}. \end{cases}$$

Therefore

$$\|V^\varepsilon(x,\cdot)\|_{L^1_z(0,1)} \leq \int_0^1 V^{\varepsilon,+}(x,z)\,dz + \int_0^1 V^{\varepsilon,-}(x,z)\,dz.$$

The right-hand side can be treated as above, which leads to

$$\|V^\varepsilon\|_{L^{\infty,1}(\Omega)} \leq C\left(\|\rho^+\|_{\ell^1(L^q)} + \|\rho^-\|_{\ell^1(L^q)}\right) \leq C\|\rho\|_{\ell^1(L^q)}.$$

Since $V_{ext} \in L^{\infty,1}(\Omega)$, we deduce from Lemmas 3.1 and A.3 that the χ_p's are bounded in $L^\infty(\Omega)$ independently of p:

$$\|\chi_p\|_{L^\infty(\Omega)} \leq C_\rho = C_1 \exp\left(C_2\,\|\rho\|_{\ell^1(L^q)}\right),$$

where the constants are independent of p and ε (and only the constant C_1 depends on $\|V_{ext}\|_{L^{\infty,1}}$). Therefore the right-hand side $R^\varepsilon[\sum_p \rho_p|\chi_p|^2]$ of the Poisson equation

(3.5) is in $L^q(\Omega)$ and its norm is bounded by $(C_\rho)^2 \|\rho\|_{\ell^1(L^q)}$. Elliptic regularity results show that $V \in W^{2,q}(\Omega)$ and finishes the proof of Theorem 3.3, item *(ii)*.

Proof of Item *(i)***.** For $\frac{d}{2} \geq \frac{2d}{d+1}$ (*i.e.* $d \geq 3$), Proposition 3.6 insures the existence of a solution. The fact that V^ε belongs to $W^{2,q}(\Omega)$ is a consequence of Item *(ii)* of Theorem 3.3.

In the case $\frac{d}{2} < \frac{2d}{d+1}$, we set $\rho_p^n = \dfrac{\rho_p}{1+\frac{1}{n}p^2|\rho_p|}$ for $n>0$ and define V^n as the corresponding solution of the regularized Schrödinger-Poisson system constructed in Proposition 3.6. It is clear that $(\rho^n) \in \ell^1(L^\infty(\omega))$ and that

$$\|\rho^n\|_{\ell^1(L^q)} \leq \|\rho\|_{\ell^1(L^q)}.$$

Item *(ii)* implies that $\|V^n\|_{W^{2,q}(\Omega)} \leq C$, where C is independent of n. Hence by Lemma B.1 we can extract a subsequence which converges as $n \to \infty$ in the $L^{\infty,q}(\Omega)$ strong topology and in $W^{2,q}(\Omega)$ weak. The inequalities (A.23) and (A.24) enable to deduce that for any fixed p the sequences ϵ_p^n and χ_p^n also converge as $n \to \infty$ respectively in $L^\infty(\omega)$ and $L^\infty(\Omega)$ (uniformly with respect to p). Therefore we can pass to the limit in (3.4)-(3.5) and the limit V is a $W^{2,q}(\Omega)$ solution of this system. The proof of Theorem 3.3 is complete. $\square$

3.3 Step 3 : Uniqueness and continuity (proof of Theorem 3.4)

Let us start with Item *(ii)*. Denoting respectively by V^ε and $\tilde{V}^\varepsilon$ two solutions corresponding to ρ and $\tilde\rho$, we have

$$-\Delta(V^\varepsilon - \tilde{V}^\varepsilon) = R^\varepsilon\left[\sum_{p\geq 1}(\rho_p - \tilde\rho_p)\,|\tilde\chi_p^\varepsilon|^2\right] + R^\varepsilon\left[\sum_{p\geq 1}\rho_p\left(|\chi_p^\varepsilon|^2 - |\tilde\chi_p^\varepsilon|^2\right)\right]. \qquad (3.8)$$

By Item *(ii)* of Theorem 3.3 and the embedding of $W^{2,q}(\Omega)$ in $C^0(\overline{\omega}, L^q(0,1))$ we have

$$\|V^\varepsilon\|_{L^{\infty,q}(\Omega)} + \|\tilde{V}^\varepsilon\|_{L^{\infty,q}(\Omega)} \leq C_{\rho,\tilde\rho,V_{ext}},$$

where $C_{\rho,\tilde\rho,V_{ext}}$ denotes a generic constant depending on $\|\rho\|_{\ell^1(L^q)}$, $\|\tilde\rho\|_{\ell^1(L^q)}$ and on $\|V_{ext}\|_{L^{\infty,1}(\Omega)}$ (uniform in ε). From (A.12), (A.24) and Lemma 3.1 we deduce that

$$\|\chi_p^\varepsilon\|_{L^\infty(\Omega)} + \|\tilde\chi_p^\varepsilon\|_{L^\infty(\Omega)} \leq C_{\rho,\tilde\rho,V_{ext}},$$

$$\|\chi_p^\varepsilon - \tilde\chi_p^\varepsilon\|_{L^\infty(\Omega)} \leq C_{\rho,\tilde\rho,V_{ext}}\,\|V^\varepsilon - \tilde{V}^\varepsilon\|_{W^{2,q}(\Omega)}.$$

Therefore

$$\|\Delta(V^\varepsilon - \tilde{V}^\varepsilon)\|_{L^q(\Omega)} \leq C_{\rho,\tilde\rho,V_{ext}}\,\|\rho - \tilde\rho\|_{\ell^1(L^q)} + C_{\rho,\tilde\rho,V_{ext}}\,\|\rho\|_{\ell^1(L^q)}\,\|V^\varepsilon - \tilde{V}^\varepsilon\|_{W^{2,q}(\Omega)}.$$

Elliptic regularity implies

$$\|V^\varepsilon - \tilde{V}^\varepsilon\|_{W^{2,q}(\Omega)} \le C_{\rho,\tilde\rho,V_{ext}} \|\rho - \tilde\rho\|_{\ell^1(L^q)} + C_{\rho,\tilde\rho,V_{ext}} \|\rho\|_{\ell^1(L^q)} \|V^\varepsilon - \tilde{V}^\varepsilon\|_{W^{2,q}(\Omega)}. \quad (3.9)$$

By noticing that under the assumptions $\|\rho\|_{\ell^1(L^q)} < \mathcal{N}$ and $\|\tilde\rho\|_{\ell^1(L^q)} < \mathcal{N}$, the quantity $C_{\rho,\tilde\rho,V_{ext}} \|\rho\|_{\ell^1(L^q)}$ tends to 0 as $\mathcal{N}$ tends to 0, one can choose $\mathcal{N}$ small enough so that quantity is smaller than 1/2. The Lipschitz estimate stated in *(ii)* finally follows from (3.9). Remark that $\mathcal{N}$ depends only on $\|V_{ext}\|_{L^{\infty,1}}$.

The first part of Item *(i)* is a direct consequence of *(ii)*. For the second part, let (ρ) be given, satisfying $\|\rho\|_{\ell^1(L^q)} \le \mathcal{N}$, and let V^ε (resp. V) be the solution of the regularized (resp. unregularized) Schrödinger-Poisson problem. We have

$$-\Delta(V^\varepsilon - V) = (R^\varepsilon - Id)\left[\sum_{p\ge1} \rho_p\, |\chi_p|^2\right] + R^\varepsilon\left[\sum_{p\ge1} \rho_p\, \left(|\chi_p^\varepsilon|^2 - |\chi_p|^2\right)\right]. \quad (3.10)$$

Since $\left\|\sum_{p\ge1} \rho_p\, |\chi_p|^2\right\|_{L^q(\Omega)} \le C\mathcal{N}$, Lemma 3.1 implies the convergence to zero (as $\varepsilon \to 0$, and in $L^q(\Omega)$) of the first term of the right-hand side of (3.10). The second term can be estimated as above by using (A.24) and Lemma 3.1:

$$\begin{aligned}
\left\|\sum_{p\ge1} \rho_p\, \left(|\chi_p^\varepsilon|^2 - |\chi_p|^2\right)\right\|_{L^q(\Omega)} &\le C\mathcal{N} \|R^\varepsilon[V^\varepsilon] - V\|_{L^{\infty,1}(\Omega)} \\
&\le C\mathcal{N}\left(\|R^\varepsilon[V^\varepsilon] - R^\varepsilon[V]\|_{L^{\infty,1}(\Omega)} + \|R^\varepsilon[V] - V\|_{L^{\infty,1}(\Omega)}\right) \\
&\le C\mathcal{N}\left(\|V^\varepsilon - V\|_{W^{2,q}(\Omega)} + \|R^\varepsilon[V] - V\|_{L^{\infty,1}(\Omega)}\right).
\end{aligned}$$

Besides, the embedding $W^{2,q}(\Omega) \subset C^0(\overline{\omega}, L^q(0,1))$ and the boundary condition $V = 0$ on $\partial\omega \times (0,1)$ imply that $\|R^\varepsilon[V] - V\|_{L^{\infty,1}(\Omega)}$ converges to 0 (apply Lemma 3.1). Therefore

$$\| - \Delta(V^\varepsilon - V)\|_{L^q} \le C\mathcal{N}\|V^\varepsilon - V\|_{W^{2,q}(\Omega)} + o(1).$$

Standard elliptic estimates and the smallness of $\mathcal{N}$ imply that V^ε converges to V in $W^{2,q}(\Omega)$ strong.

It remains to prove *(iii)*. Since $\|\rho\|_{\ell^1(L^q)} \le \mathcal{N}$, Item *(i)* applies and yields the boundedness of V^ε in $L^{\infty,q}(\Omega)$. Consequently, (A.12) implies the uniform estimates

$$\|\chi_p^\varepsilon\|^2_{L^\infty(\Omega)} \le C \quad ; \quad \|\partial_z \chi_p^\varepsilon\|^2_{L^\infty(\Omega)} \le C\, p^2.$$

The second inequality gives the estimate of $\mathcal{E}^\varepsilon_{kin,z}$ (first part of (3.6)) while the estimate of $\mathcal{E}^\varepsilon_{pot}$ is obtained by multiplying (3.5) by V^ε and integrating on Ω,

$$\begin{aligned}
\iint |\nabla_{x,z} V^\varepsilon|^2\, dx\, dz &= \iint V^\varepsilon\, R^\varepsilon\left[\sum_{p\ge1} \rho_p\, \left|\chi_p^\varepsilon\right|^2\right] dx\, dz \\
&\le \|V^\varepsilon\|_{L^{\infty,1}(\Omega)} \sup_{p\ge1} \|\chi_p^\varepsilon\|^2_{L^\infty(\Omega)} \|\rho\|_{\ell^1(L^1)} \\
&\le C\, \|\rho\|_{\ell^1(L^1)}.
\end{aligned}$$

□

4 Analysis of the stationary Vlasov-Schrödinger-Poisson problem

This section is devoted to the proof of Theorem 2.2. It is independent of Section 3. The proof relies on the application of Schauder fixed point theorem for a regularization of the problem and uses the supersolution technique developed by Poupaud in [43]. More precisely, let ε and λ be two positive regularization parameters that we shall let tend to zero. Consider the mapping $S_{\lambda,\varepsilon}$ defined on $L^\infty(\Omega)$ in the following way : for a given potential $V \in L^\infty(\Omega)$, let (ϵ_p, χ_p) be the solution of

$$\begin{cases} -\dfrac{1}{2}\,\partial_{zz}\chi_p(x;z) + (R^\varepsilon[V] + V_{ext})\chi_p(x;z) = \epsilon_p(x)\chi_p(x;z), \\ \chi_p(x,\cdot) \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,, \end{cases} \tag{4.1}$$

where the regularization operator R^ε is defined in (3.3). Then, compute f_p by solving

$$\begin{cases} \lambda f_p + v\cdot\nabla_x f_p - \nabla_x\epsilon_p\cdot\nabla_v f_p = 0, \\ \gamma^- f_p = \alpha g_p + (1-\alpha)\mathcal{R}_p(\gamma^+ f_p) \quad \text{on } \Sigma_-\,. \end{cases} \tag{4.2}$$

Finally, define $V^* = S_{\lambda,\varepsilon}(V)$ as the unique solution of

$$\begin{cases} -\Delta V^* = R^\varepsilon\left[\displaystyle\sum_{p\geq 1}\left(\int_{\mathbb{R}^d} f_p\,dv\right)|\chi_p|^2\right], \\ V^* = 0 \quad \text{on } \partial\omega\times(0,1), \\ \partial_z V^* = 0 \quad \text{on } \omega\times\{0\}\ \cup\ \omega\times\{1\}. \end{cases} \tag{4.3}$$

The mapping $S_{\lambda,\varepsilon}$ is clearly uniquely defined. Indeed, the only thing to be checked is the existence and uniqueness of solutions of (4.2). This is a consequence of the fact that ϵ_p is in $C^2(\overline{\omega})$ (by Lemma A.6, the ϵ_p have the same regularity as the potential, see also Lemma 4.4 at the end of this section) and to the following

Lemma 4.1 *Assume that $\epsilon_p \in C^2(\overline{\omega})$. Under the assumptions* **(HS-1)** *and* **(HS-3)**, (4.2) *admits a unique weak solution $f_p \in L^1(\omega\times\mathbb{R}^d)$. Moreover, we have*

$$0 \leq f_p(x,v) \leq G_p\left(\frac{v^2}{2} + \epsilon_p(x)\right).$$

Proof. Uniqueness of solutions follows from the fact that $\lambda > 0$, $\alpha > 0$ and from the following Darrozès-Guiraud inequality: for any convex nonnegative and continuous function J and for $x \in \partial\omega$ we have

$$\int_{v\cdot\nu(x)<0} J\left(\mathcal{R}_p(f)\right)(x,v)\,|v\cdot\nu(x)|dv \leq \int_{v\cdot\nu(x)>0} J(f)(x,v)\,|v\cdot\nu(x)|dv.$$

Indeed, if f is the difference of two solutions, multiplying the Vlasov equation for f by $sgn(f)$ and integrating with respect to (x, v), one obtains

$$\begin{aligned}
\lambda \int_{\omega\times\mathbb{R}^d} |f|\, dx\, dv &= \int_{\Sigma_-} |\gamma^- f|\, d\Sigma - \int_{\Sigma_+} |\gamma^+ f| d\Sigma \\
&= (1-\alpha) \int_{\Sigma_+} |\mathcal{R}_p(\gamma^+ f)|\, d\Sigma - \int_{\Sigma_+} |\gamma^+ f| d\Sigma \\
&\leq 0.
\end{aligned}$$

The existence of solutions can be proved in the same spirit as in [43] (the index p is skipped in this proof for simplicity):

$$f = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f_k$$

where the limit is taken in the weak sense and f_k is defined by

$$\begin{cases} \lambda f_k + v\cdot\nabla_x f_k - \nabla_x \epsilon \cdot \nabla_v f_k = 0, \\ \gamma^- f_k = \alpha g + (1-\alpha)\mathcal{R}(\gamma^+ f_{k-1}), \end{cases}$$

with $f_1(x, v) = G(v^2/2 + \epsilon(x))$. The function f exists, since it is readily seen by induction that $0 \leq f_k \leq G(v^2/2+\epsilon(x))$ and is obviously a solution of (4.2) satisfying the supersolution estimate $0 \leq f \leq G(v^2/2 + \epsilon(x))$. □

The mapping $S_{\lambda,\varepsilon}$ satisfies the following properties:

Lemma 4.2 *Let $B_+^r = \{V \in L^\infty(\Omega) \ : \ 0 \leq V(x, z) \leq r \ a.e.\}$. Then there exists $R > 0$ independent of λ and ε such that*

$$S_{\lambda,\varepsilon}(B_+^R) \subset B_+^R.$$

Moreover, for all $q \in [1, +\infty)$, there exists a constant C_q independent of λ and ε such that

$$S_{\lambda,\varepsilon}(B_+^R) \subset \{V \in W^{2,q}(\Omega) \ : \quad \|V\|_{W^{2,q}(\Omega)} \leq C_q\}.$$

Proof. Let $V \in B_+^R$. The supersolution estimate shows that

$$0 \leq \rho_p(x) \leq |S_{d-1}| \int_0^{+\infty} |u|^{d-1} G_p\left(\frac{u^2}{2} + \epsilon_p(x)\right) du,$$

where $|S_{d-1}|$ is the measure of the unit sphere. Since $V + V_{ext} \geq 0$, it follows from Hypothesis **(HS-3)** and from (A.6) that $\rho_p(x) \leq C\, M_p$ where

$$M_p = \int_0^{+\infty} |u|^{d-1} G_p\left(\frac{u^2}{2} + \frac{\pi^2 p^2}{2}\right) du.$$

Applying (A.10) with $r = +\infty$ yields

$$0 \le -\Delta V^* \le \sum_{p \ge 1} p M_p + R^{1/2} \sum_{p \ge 1} M_p$$

The first inequality implies the positivity of V^*, while the second one yields by elliptic regularity

$$\|V^*\|_{W^{2,q}} \le C_q (1 + R^{1/2}) \qquad \forall q < +\infty. \tag{4.4}$$

which implies the inequality $\|V^*\|_{L^\infty} \le C(1 + R^{1/2})$ since $W^{2,q}$ is embedded in L^∞ for q large enough. It is now enough to choose R such that $R \ge C(1 + R^{1/2})$ which shows the first part of the lemma. The second part follows from (4.4). □

Remark 4.3 *The hypothesis $V_{ext} \in C^2(\overline{\omega})$ is not necessary. It is enough to assume that $V_{ext} \in L^{\infty,1}(\Omega)$ and that $\nabla_x V_{ext} \in L^1(\Omega)$. With these hypotheses, one has to mollify V_{ext} in the regularized problem. The above hypotheses insure that $\nabla_x \epsilon_p \in L^1(\omega)$. This is enough to pass weakly to the limit in the nonlinear terms of the Vlasov equation $\nabla_x \epsilon_p \cdot \nabla_v f_p$ since f_p is bounded in L^∞. Moreover Item (ii) of Theorem* 3.3 *gives a uniform bound of V in any $W^{2,q}(\Omega)$, which enables to pass to the limit in the Schrödinger-Poisson system.*

The above lemma shows the existence of a convex bounded set of $L^\infty(\Omega)$ which is let invariant by $S_{\lambda,\varepsilon}$ and also shows the compactness of $S_{\lambda,\varepsilon}$. The continuity of $S_{\lambda,\varepsilon}$ is easily obtained (the details are left to the reader). Hence, $S_{\lambda,\varepsilon}$ admits a fixed point $V^{\lambda,\varepsilon}$. The passage to the limit $\lambda, \varepsilon \to 0$ can be done without difficulty since $V^{\lambda,\varepsilon}$ is bounded uniformly in $W^{2,q}$ for all $q < +\infty$. This shows the existence of solutions of the unmodified stationary Vlasov-Schrödinger-Poisson problem (2.5)–(2.7).

To complete the proof of Theorem 2.2, it remains to explain how the regularity of the eigenvalues ϵ_p and eigenvectors χ_p can be deduced from the regularity of the potential V. This is the object of the following Lemma, which uses the notations of the Appendix A:

Lemma 4.4 *Let $q \in (\max(1, d/2), +\infty]$. Assume that $V \in W^{2,q}(\Omega)$. Then $\epsilon_p[V] \in W^{2,q}(\omega)$ and $\chi_p[V] \in W^{2,q}(\Omega)$.*

Proof. Since $q > \max(1, d/2)$, Lemma B.1 implies that $W^{1,q}(\Omega) \subset L^{2q,q}(\Omega)$. Therefore $\nabla_x V \in L^{2q,q}(\Omega)$. Besides, Lemma B.1 *(iv)* implies that $V \in L^{\infty,q}(\Omega)$. Consequently, (A.11), (A.16) and (A.20) lead to $\epsilon_p \in W^{2,q}(\omega)$. Similarly for χ_p (A.12), (A.18), (A.20) and (A.19) show resp. that χ_p, $\nabla_x \chi_p$ and the second derivatives $\frac{\partial^2}{\partial x_i \partial x_j}$ or $\frac{\partial^2}{\partial x_i \partial z}$ of χ_p are in $L^q(\Omega)$. We conclude the proof by using directly the equation (A.3) to estimate $\frac{\partial^2}{\partial z^2}\chi_p$. □

5 Analysis of the time-dependent Vlasov-Schrödinger-Poisson problem

The aim of this section is the proof of Theorem 2.3. The strategy relies on a fixed point argument, as in the stationary case, but is more complicated due to the time-dependence. Indeed, the Poisson equation provides compactness with respect to position variables, as seen in the stationary case, but not with respect to time. Therefore, we shall use the averaging lemmas of the Vlasov equation, coupled to the results of Section 3 (Theorems 3.3 and 3.4), which requires the smallness of the distribution functions.

5.1 The energy estimate

We present here some *a priori* estimates which are satisfied by solutions of the Vlasov-Schrödinger-Poisson system (2.9)-(2.11).

Proposition 5.1 (Energy estimate). *Let $(V, (\epsilon_p, \chi_p, f_p)_{p\geq 1})$ be a solution of (2.9)-(2.11). Let us define total energy*

$$\mathcal{E}_{tot}(t) = \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t) \tag{5.1}$$

where the kinetic energy and the potential energy are respectively defined by

$$\mathcal{E}_{kin}(t) = \sum_{p\geq 1} \iint \frac{v^2}{2} f_p \, dx\, dv + \sum_{p\geq 1} \iiint \frac{1}{2} |\partial_z \chi_p|^2 f_p \, dx\, dz\, dv \tag{5.2}$$

$$\mathcal{E}_{pot}(t) = \iint n V_{ext} \, dx\, dz + \iint \frac{1}{2} |\nabla_{x,z} V|^2 \, dx\, dz. \tag{5.3}$$

Then, we have

$$\begin{aligned}
\mathcal{E}_{tot}(t) &= \mathcal{E}_{tot}(0) + \int_0^t \iint_\Omega n\, \partial_t V_{ext} \, dx\, dz\, ds - \sum_{p\geq 1} \int_0^t \iint_\Sigma \left(\frac{v^2}{2} + \epsilon_p \right) f_p \, v\cdot\nu \, d\sigma\, dv\, ds \\
&\leq \mathcal{E}_{tot}(0) + \int_0^t \|n(s)\|_{L^1} \, \|[\partial_t V_{ext}(s)]^+\|_{L^\infty} \, ds + \alpha \sum_{p\geq 1} \int_0^t \iint_{\Sigma_-} \left(\frac{v^2}{2} + \epsilon_p \right) g_p \, d\Sigma\, ds.
\end{aligned} \tag{5.4}$$

Moreover, the following estimate holds

$$\|n(t)\|_{L^1} \leq \|n(0)\|_{L^1} + \alpha \sum_{p\geq 1} \int_0^t \iint_{\Sigma_-} g_p \, d\Sigma\, ds. \tag{5.5}$$

Proof. The integration of (2.9) with respect to v yields the charge conservation equation

$$\partial_t \rho_p + \mathrm{div}_x\, j_p = 0. \tag{5.6}$$

Multiplying (2.9) by $\frac{v^2}{2}$ and integrating with respect to x and v provides, after some integrations by parts in v and x and the use of (5.6):

$$\frac{d}{dt}\iint \left(\frac{v^2}{2} + \epsilon_p\right) f_p\, dx\, dv - \iint f_p\, \partial_t \epsilon_p\, dx\, dv = -\iint_\Sigma \left(\frac{v^2}{2} + \epsilon_p\right) f_p\, v\cdot\nu\, d\sigma\, dv. \tag{5.7}$$

Identity (A.4) of Appendix A gives

$$\sum_{p\geq 1}\iint \epsilon_p\, f_p\, dx\, dv = \sum_{p\geq 1}\iiint \frac{1}{2}\,|\partial_z \chi_p|^2\, f_p\, dx\, dz\, dv + \iint (V + V_{ext}) n\, dx\, dz$$

and (A.15) implies

$$-\sum_{p\geq 1}\iint f_p\, \partial_t \epsilon_p\, dx\, dv = -\iint n\, \partial_t (V + V_{ext})\, dx\, dz.$$

Consequently, after a summation on p, (5.7) becomes

$$\begin{aligned}\frac{d}{dt}&\left(\sum_{p\geq 1}\iint \frac{v^2}{2} f_p\, dx\, dv + \sum_{p\geq 1}\iiint \frac{1}{2}\,|\partial_z \chi_p|^2\, f_p\, dx\, dz\, dv + \iint n\, V_{ext}\, dx\, dz\right)\\ &+ \iint V\, \partial_t n\, dx\, dz = \iint n\, \partial_t V_{ext}\, dx\, dz - \sum_{p\geq 1}\iint_\Sigma \left(\frac{v^2}{2} + \epsilon_p\right) f_p\, v\cdot\nu\, d\sigma\, dv.\end{aligned}$$

Besides, it is readily seen from the Poisson equation (2.11) that

$$\iint V\, \partial_t n\, dx\, dz = \frac{d}{dt}\iint \frac{1}{2}\,|\nabla_{x,z} V|^2\, dx\, dz.$$

This shows that

$$\frac{d\mathcal{E}_{tot}}{dt} = \iint_\Omega n\, \partial_t V_{ext}\, dx\, dz - \sum_{p\geq 1}\iint_\Sigma \left(\frac{v^2}{2} + \epsilon_p\right) f_p\, v\cdot\nu\, d\sigma\, dv.$$

Inequality (5.4) is a consequence of the following identity which is satisfied for any real-valued function ψ :

$$\iint_{\Sigma_-} \psi(|v|^2)\, \mathcal{R}_p(\gamma^+ f)\, d\Sigma = \iint_{\Sigma_+} \psi(|v|^2)\, (\gamma^+ f)\, d\Sigma.$$

Finally, (5.5) is obtained by integrating the Vlasov equation with respect to t, x, v and by applying the above identity with $\psi \equiv 1$. □

5.2 The linear Vlasov equation

Let us now give some results about the collection of Vlasov equations (2.9). We shall assume that the force fields $F_p = -\nabla_p \epsilon_p$ are known and write the equations indexed by p

$$\begin{cases} \partial_t f_p + v \cdot \nabla_x f_p + F_p \cdot \nabla_v f_p = 0, \\ \gamma^- f_p = \alpha g_p + (1-\alpha)\mathcal{R}_p(\gamma^+ f_p) \quad \text{on } (0,T) \times \Sigma_- \,, \\ f_p(0,x,v) = f_{p,0}(x,v). \end{cases} \tag{5.8}$$

The following lemma states the existence and uniqueness of the weak solution for each Vlasov equation (5.8):

Lemma 5.2 *Assume that the initial and boundary data $f_{p,0}$ and g_p satisfy*

$$f_{p,0} \geq 0, \quad (1+v^2) f_{p,0} \in L^1(\omega \times \mathbb{R}^d), \quad f_{p,0} \in L^\infty(\omega \times \mathbb{R}^d),$$

$$g_p \geq 0, \quad (1+v^2) g_p \in L^1((0,T), L^1(\Sigma_-, d\Sigma)), \quad g_p \in L^\infty((0,T) \times \Sigma_-).$$

Assume that $F_p \in L^1((0,T), W^{1,1}(\omega) \cap L^\infty(\omega))$. Then (5.8) admits a unique weak solution $f_p \in L^\infty((0,T),\, L^1 \cap L^\infty(\omega \times \mathbb{R}^d))$. Defining the kinetic energy by

$$\mathcal{E}_{kin,p}(t) = \iint_{\omega \times \mathbb{R}^d} \frac{v^2}{2}\, f_p(t,x,v)\, dx\, dv,$$

the following estimates are satisfied by the solution f_p:

$$0 \leq f_p(t,x,v) \leq \max(\|g_p\|_{L^\infty}, \|f_{p,0}\|_{L^\infty}) \quad a.e., \tag{5.9}$$

$$\iint_{\omega \times \mathbb{R}^d} f_p(t,x,v)\, dx\, dv \leq \iint_{\omega \times \mathbb{R}^d} f_{p,0}(x,v)\, dx\, dv + \alpha \int_0^t \iint_{\Sigma_-} g_p(s,x,v)\, d\Sigma\, ds, \tag{5.10}$$

$$\mathcal{E}_{kin,p}(t) \leq \mathcal{E}_{kin,p}(0) + \alpha \int_0^t \iint_{\Sigma_-} \frac{v^2}{2}\, g_p(s,x,v)\, d\Sigma\, ds + C\left(\Phi(t) + \Phi(t)^{d+2}\right), \tag{5.11}$$

where

$$\Phi(t) = \|f_p\|^{1/(d+2)}_{L^\infty((0,t)\times\omega\times\mathbb{R}^d)} \int_0^t \|F_p(s,\cdot)\|_{L^\infty}\, ds$$

and C is a constant independent of p and of the data.

Proof. In this proof the index p is only a parameter and is omitted for notational simplicity. The construction of a solution can be done in the same way as in the proof of Lemma 4.1 :

$$f = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f_k$$

where f_k is defined by

$$\left\{\begin{array}{l}\partial_t f_k + v\cdot\nabla_x f_k + F\cdot\nabla_v f_k = 0,\\[2mm] \gamma^- f_k = \alpha g + (1-\alpha)\mathcal{R}(\gamma^+ f_{k-1}),\\[2mm] f_k(t=0) = f_{p,0}.\end{array}\right.$$

The regularity of the force field ensures the existence and uniqueness of f_k [36]. The L^∞ bound (5.9) is satisfied by all the f_k and is therefore satisfied by the solution f. Estimate (5.10) can be obtained after a simple integration of (5.8) with respect to t, x and v. Estimate (5.11) is obtained as follows : multiplying (5.8) by $\frac{v^2}{2}$ then integrating on $\omega\times\mathbb{R}^d$, one obtains

$$\frac{d\mathcal{E}_{kin}}{dt} = -\iint_\Sigma \frac{v^2}{2} f(t,x,v)\, v\cdot\nu \, d\sigma dv + \int_\omega F(t,x)\cdot j(t,x)\,dx. \tag{5.12}$$

Like in the proof of Proposition 5.1, we have

$$-\iint_\Sigma \frac{v^2}{2} f(t,x,v)\, v\cdot\nu\, d\sigma dv = \alpha \iint_{\Sigma_-} \frac{v^2}{2} g(t,x,v)\, d\Sigma.$$

Besides, the classical interpolation result [13]

$$|j(t,x)| \le C\,\|f\|_{L^\infty}^{1/(d+2)} \left(\int \frac{v^2}{2}\, f(t,x,v)\,dv\right)^{(d+1)/(d+2)},$$

and Jensen's inequality lead to

$$\int_\omega F(t,x)\cdot j(t,x)\,dx \le C\,\|F(t,\cdot)\|_{L^\infty}\,\|f\|_{L^\infty}^{1/(d+2)} \left(\iint \frac{v^2}{2}\, f(t,x,v)\,dv\,dx\right)^{(d+1)/(d+2)},$$

(the domain ω is bounded). Inserting this inequality in (5.12), one obtains

$$\frac{d\mathcal{E}_{kin}}{dt}(t) \le C\,\|F(t,\cdot)\|_{L^\infty}\,\|f\|_{L^\infty}^{1/(d+2)} (\mathcal{E}_{kin}(t))^{(d+1)/(d+2)} + \alpha \iint_{\Sigma_-} \frac{v^2}{2} g(t,x,v)\,d\Sigma.$$

This leads after a Gronwall argument, to

$$\begin{aligned}(\mathcal{E}_{kin}(t))^{1/(d+2)} \le\ & \left(\mathcal{E}_{kin}(0) + \alpha\int_0^t \iint_{\Sigma_-} \frac{v^2}{2}\, g(s,x,v)\,d\Sigma\,ds\right)^{1/(d+2)} \\ & + C\,\|f\|_{L^\infty}^{1/(d+2)} \int_0^t \|F(s,\cdot)\|_{L^\infty}\,ds,\end{aligned}$$

which, raised to the power $d+2$, leads to (5.11) (the inequality $(1+h)^p \le 1 + C(h+h^p)$ is used). □

The following interpolation inequality will be used in the remainder of this section:

Lemma 5.3 *Assume (f_p) lies in $\ell^1(L^\infty(\mathcal{O}\times\mathbb{R}^d))$ and $(v^2 f_p)\in\ell^1(L^1(\mathcal{O}\times\mathbb{R}^d))$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^N$. Then we have*

$$\sum_{p\geq 1}\|\rho_p\|_{L^{(d+2)/d}(\mathcal{O})}\leq C\left(\sum_{p\geq 1}\|f_p\|_{L^\infty(\mathcal{O}\times\mathbb{R}^d)}\right)^{\frac{2}{d+2}}\left(\sum_{p\geq 1}\|v^2 f_p\|_{L^1(\mathcal{O}\times\mathbb{R}^d)}\right)^{\frac{d}{d+2}}, \quad (5.13)$$

$$\sum_{p\geq 1}\|j_p\|_{L^{(d+2)/(d+1)}(\mathcal{O})}\leq C\left(\sum_{p\geq 1}\|f_p\|_{L^\infty(\mathcal{O}\times\mathbb{R}^d)}\right)^{\frac{1}{d+2}}\left(\sum_{p\geq 1}\|v^2 f_p\|_{L^1(\mathcal{O}\times\mathbb{R}^d)}\right)^{\frac{d+1}{d+2}}. \quad (5.14)$$

Proof. The two following interpolation estimates are standard and can be found for instance in [13]:

$$\|\rho_p\|_{L^{(d+2)/d}(\mathcal{O})}\leq C\|f_p\|_{L^\infty}^{2/(d+2)}\left(\|v^2 f_p\|_{L^1(\mathcal{O}\times\mathbb{R}^d)}\right)^{d/(d+2)}$$

$$\|j_p\|_{L^{(d+2)/(d+1)}(\mathcal{O})}\leq C\|f_p\|_{L^\infty}^{1/(d+2)}\left(\|v^2 f_p\|_{L^1(\mathcal{O}\times\mathbb{R}^d)}\right)^{(d+1)/(d+2)}.$$

Then (5.13) and (5.14) can be deduced easily, after a Hölder inequality. □

Corollary 5.4 *Let $d\leq 3$ and let $f_p(x,v)$ be such that $f_p\geq 0$. There exists a constant $\mathcal{E}$ depending only on V_{ext} (and independent of ε) such that if*

$$\left(\sum_{p\geq 1}\|f_p\|_{L^\infty}\right)^{2/d}\left(\sum_{p\geq 1}\iint\frac{v^2}{2}\,f_p\,dx\,dv\right)<\mathcal{E} \quad (5.15)$$

then the collection of density functions defined by $\rho_p=\int f_p\,dv$ satisfy $\|\rho\|_{\ell^1(L^{(d+2)/d})}<\mathcal{N}$, where $\mathcal{N}$ is the same constant as in Theorem 3.4. *Consequently, the Schrödinger-Poisson problem* (3.1)-(3.2) *as well as the regularized Schrödinger-Poisson problem* (3.4)-(3.5) *admit unique solutions.*

Proof. This lemma is an immediate consequence of (5.13). To apply Theorem 3.4, we only have to check that $q=\frac{d+2}{d}$ satisfies the condition $q>\max(1,\frac{d}{2})$. This is true if (and only if) $d\leq 3$. □

It is well-known that the average quantities of f with respect to the velocity satisfy compactness properties. The following Lemma makes them precise in the case of series (f_p):

Lemma 5.5 *Let $F_p^n\in L^1((0,T)\times\omega)$ for any integer p and n. Let (f_p^n) be a sequence of solutions of* (5.8) *such that for all $n\in\mathbb{N}$*

$$\sum_{p\geq 1}\|f_p^n\|_{L^\infty((0,T)\times\omega\times\mathbb{R}^d)}\leq C\quad;\quad\sum_{p\geq 1}\|(v^2+p^2)f_p^n\|_{L^1((0,T)\times\omega\times\mathbb{R}^d)}\leq C,$$

with uniform bounds with respect to n. Then, the sequence $(\rho_p^n)_{p\geq 1}=(\int f_p^n\,dv)_{p\geq 1}$ is compact (with respect to the index n) in the $\ell^1(L^q((0,T)\times\omega))$ topology for any $q<\frac{d+2}{d}$.

Proof. Since Lemma 5.3 yields the boundedness of (ρ^n) in $\ell^1(L^{(d+2)/2}((0,T)\times\omega))$, it is enough to prove the compactness of (ρ^n) in $\ell^1(L^1((0,T)\times\omega))$.

The first step of this proof is the standard mean compactness result from [26, 27, 21] (more precisely, we refer to Theorem 5.13 of the review paper [13]): for any function $\psi\in C_c^\infty(\mathbb{R}^d)$ and for any fixed p, the sequence indexed by n

$$\int_{\mathbb{R}^d} f_p^n(t,x,v)\psi(v)\,dv$$

is compact in $L^1((0,T)\times\omega)$.

Let $\psi\in C_c^\infty(\mathbb{R}^d)$ be a cut-off function such that $0\le\psi\le 1$ and $\psi(v)=1$ if $|v|\le 1$. For any integer $R>0$ the particle density can be written

$$\rho_p^n=\rho_p^{n,R}+r_p^{n,R}$$

with

$$\rho_p^{n,R}=\mathbb{1}_{p\le R}\int_{\mathbb{R}^d} f_p^n(t,x,v)\psi(v/R)\,dv$$

and

$$r_p^{n,R}=\mathbb{1}_{p>R}\int_{\mathbb{R}^d} f_p^n(t,x,v)\psi(v/R)\,dv+\int_{\mathbb{R}^d} f_p^n(t,x,v)\left(1-\psi(v/R)\right)\,dv.$$

With a diagonal extraction procedure, for any $R\in\mathbb{N}^*$ (a subsequence of) the sequence $(\rho^{n,R})$ converges in $\ell^1(L^1((0,T)\times\omega))$ as $n\to+\infty$. In order to bound the remainder $(r^{n,R})$ we make use of the bound of $(v^2+p^2)f_p^n$ in $\ell^1(L^1)$:

$$\begin{aligned}\sum_{p\ge1}\left\|r_p^{n,R}\right\|_{\ell^1(L^1((0,T)\times\omega))}&\le\sum_{p>R}\int_{\mathbb{R}^d}\frac{p^2}{R^2}f_p^n(t,x,v)\psi(v/R)\,dv\\&\quad+\sum_{p\ge1}\int_{|v|\ge R}\frac{v^2}{R^2}f_p^n(t,x,v)\left(1-\psi(v/R)\right)\,dv\\&\le\frac{1}{R^2}\sum_{p\ge1}\|(v^2+p^2)f_p^n\|_{L^1((0,T)\times\omega\times\mathbb{R}^d)}\le\frac{C}{R^2}.\end{aligned}$$

Thus $(r^{n,R})$ is small uniformly with respect to n in the $\ell^1(L^1((0,T)\times\omega))$ topology when R is large. This is enough to deduce that the sequence (ρ^n) is a Cauchy sequence in this topology. $\square$

We end this section by a stability result:

Lemma 5.6 *Let $F_p^n\in L^1((0,T)\times\omega)$ be a collection of force fields and let f_p^n be the corresponding sequence of weak solutions of* (5.8). *(i) If for all $p\ge 1$*

$$\begin{cases} F_p^n\overset{n\to\infty}{\longrightarrow}F_p \quad \textit{in}\ \ L^1_{t,x}\\[2ex] f_p^n\overset{n\to\infty}{\rightharpoonup}f_p \quad \textit{in}\ L^\infty_{t,x,v}\ \textit{weak}\ * \end{cases}\tag{5.16}$$

then the limit f_p is a weak solution of (5.8) *with the force field F_p.*
(ii) If moreover we have

$$\sum_{p\geq 1} \|f_p^n\|_{L^\infty_{t,x,v}} + \sum_{p\geq 1} \|(v^2+p^2) f_p^n\|_{L^1((0,T)\times\omega\times\mathbb{R}^d)} \leq C, \tag{5.17}$$

where C is a constant independent of n, then for any $q < \frac{d+2}{d}$

$$(\rho^n) = \left(\int f_p^n\, dv\right)_{p\geq 1} \quad \overset{n\to\infty}{\longrightarrow} \quad (\rho) = \left(\int f_p\, dv\right)_{p\geq 1} \qquad \textit{in } \ell^1(L^q((0,T)\times\omega)).$$

Proof. Part *(i)* of this Lemma can be directly obtained from the weak formulation of (5.8). Remark that if f_p^n verifies (5.17) then the limit f_p also verifies (5.17). To prove *(ii)*, we use Lemma 5.5. After extraction of a subsequence, $(\rho_p^n)_{p\geq 1}$ converges in $\ell^1(L^q((0,T)\times\omega))$ as $n\to\infty$. It remains to identify its limit. Let $\phi \in C_c^\infty(]0,T[\times\Omega)$. Denoting $\rho_p = \int f_p\, dv$, we have

$$\begin{aligned}\sum_{p\geq 1} \iint (\rho_p^n - \rho_p)\,\phi\, dt\, dx &= \sum_{p\geq 1} \iiint_{|v|<R} (f_p^n - f_p)\,\phi\, dt\, dx\, dv \\ &\quad + \sum_{p\geq 1} \iiint_{|v|>R} f_p^n\,\phi\, dt\, dx\, dv - \sum_{p\geq 1} \iiint_{|v|>R} f_p\,\phi\, dt\, dx\, dv.\end{aligned}$$

For any fixed $R>0$, the first term converges to 0 thanks to (5.16). The second term can be estimated as follows (the third term can be estimated similarly):

$$\begin{aligned}\left|\sum_{p\geq 1} \iiint_{|v|>R} f_p^n\,\phi\, dt\, dx\, dv\right| &\leq \frac{1}{R^2} \sum_{p\geq 0} \iiint v^2\, f_p^n\, |\phi|\, dt\, dx\, dv \\ &\leq \frac{1}{R^2} \max(|\phi|) \sum_{p\geq 0} \iiint v^2\, f_p^n\, dt\, dx\, dv \leq \frac{C}{R^2}\end{aligned}$$

and can be made arbitrary small, independently of n, by choosing R large enough. Hence the limit of (ρ^n) is (ρ). The limit being unique, all the sequence converges. □

5.3 Weak solutions for the VSP problem

In order to prove the existence of solutions for the Vlasov-Schrödinger-Poisson system (2.9)-(2.11), we shall consider the system of Vlasov equations coupled to the regularized Schrödinger-Poisson system:

$$\begin{cases} \partial_t f_p^\varepsilon + v\cdot\nabla_x f_p^\varepsilon - \nabla_x \epsilon_p^\varepsilon \cdot \nabla_v f_p^\varepsilon = 0, \\ \gamma^- f_p^\varepsilon = \alpha g_p + (1-\alpha)\mathcal{R}_p(\gamma^+ f_p^\varepsilon), \quad \text{on } (0,T)\times\Sigma_-\,, \\ f_p^\varepsilon(0,x,v) = f_{p,0}(x,v), \end{cases} \tag{5.18}$$

$$
\begin{cases}
-\dfrac{1}{2}\,\partial_{zz}\chi_p^\varepsilon + (R^\varepsilon[V^\varepsilon] + V_{ext})\chi_p^\varepsilon = \epsilon_p^\varepsilon \chi_p^\varepsilon\,, \\
\chi_p^\varepsilon(t,x,\cdot) \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p^\varepsilon\,\chi_q^\varepsilon\,dz = \delta_{pq}\,,
\end{cases}
\tag{5.19}
$$

$$
\begin{cases}
-\Delta V^\varepsilon = n^\varepsilon = R^\varepsilon\left[\displaystyle\sum_{p\geq 1}\left(\int_{\mathbb{R}^d} f_p^\varepsilon\,dv\right)\,\left|\chi_p^\varepsilon\right|^2\right], \\
V^\varepsilon = 0 \quad \text{on } \partial\omega \times (0,1), \\
\partial_z V^\varepsilon = 0 \quad \text{on } \omega\times\{0\}\ \cup\ \omega\times\{1\},
\end{cases}
\tag{5.20}
$$

where $\varepsilon > 0$ and R^ε is the operator defined by (3.3).

We first prove the following Proposition:

Proposition 5.7 (Energy estimate for the regularized system) *Let $(V^\varepsilon, (\epsilon_p^\varepsilon, \chi_p^\varepsilon, f^\varepsilon)_{p\geq 1})$ be a solution of (5.18)–(5.20). Let us define the kinetic energy and the potential energy by*

$$
\mathcal{E}_{kin}^\varepsilon(t) = \sum_{p\geq 1}\iint \frac{v^2}{2} f_p^\varepsilon\,dx\,dv + \sum_{p\geq 1}\iiint \frac{1}{2}\,|\partial_z \chi_p^\varepsilon|^2\, f_p^\varepsilon\,dx\,dz\,dv \tag{5.21}
$$

$$
\mathcal{E}_{pot}^\varepsilon(t) = \iint n^\varepsilon V_{ext}\,dx\,dz + \iint \frac{1}{2}\,|\nabla_{x,z}V^\varepsilon|^2\,dx\,dz \tag{5.22}
$$

and the total energy by

$$
\mathcal{E}_{tot}^\varepsilon(t) = \mathcal{E}_{kin}^\varepsilon(t) + \mathcal{E}_{pot}^\varepsilon(t). \tag{5.23}
$$

Then for $\varepsilon \in [0,1]$ we have

$$
\begin{aligned}
\mathcal{E}_{tot}^\varepsilon(t) &= \mathcal{E}_{tot}^\varepsilon(0) + \int_0^t \iint_\Omega n^\varepsilon\,\partial_t V_{ext}\,dx\,dz\,ds - \sum_{p\geq 1}\int_0^t \iint_\Sigma \left(\frac{v^2}{2} + \epsilon_p^\varepsilon\right) f_p^\varepsilon\, v\cdot\,\nu\,d\sigma\,dv\,ds \\
&\leq \mathcal{E}_{tot}^\varepsilon(0) + \int_0^t \|n^\varepsilon(s)\|_{L^1}\,\|[\partial_t V_{ext}(s)]^+\|_{L^\infty}\,ds + \alpha\sum_{p\geq 1}\int_0^t \iint_{\Sigma_-}\left(\frac{v^2}{2} + \epsilon_p^\varepsilon\right) g_p\,d\Sigma\,ds.
\end{aligned}
\tag{5.24}
$$

Moreover, we have

$$
\|n^\varepsilon(t)\|_{L^1} \leq \|n^\varepsilon(0)\|_{L^1} + \alpha\sum_{p\geq 1}\int_0^t \iint_{\Sigma_-} g_p\,d\Sigma\,ds. \tag{5.25}
$$

Proof. The proof is identical to the proof of Proposition 5.1. One has to notice that

$$
-\sum_{p\geq 1}\iint f_p^\varepsilon\,\partial_t\epsilon_p^\varepsilon\,dx\,dv = -\iiint \sum_{p\geq 1} f_p^\varepsilon\,|\chi_p^\varepsilon|^2\,\partial_t(R^\varepsilon[V^\varepsilon] + V_{ext})\,dx\,dz\,dv
$$

and that

$$\iint n^\varepsilon R^\varepsilon[V^\varepsilon]\,dx\,dz = \iint R^\varepsilon[n^\varepsilon]V^\varepsilon\,dx\,dz = \iint |\nabla_{x,z}V^\varepsilon|^2\,dx\,dz.$$

$\square$

Corollary 5.8 *Let $T>0$ and let M_0, E_0, M_b and E_b be defined by* (2.12)-(2.13)*. There exists a constant $\overline{\mathcal{E}}$ depending only on $\|V_{ext}(0)\|_{C^2}$ and on $\int_0^T \|[\partial_t V_{ext}]^+\|_{L^\infty(\Omega)}\,dt$ such that if the data satisfy*

$$(M_0+M_b)^{2/d}(E_0+E_b) < \overline{\mathcal{E}} \tag{5.26}$$

then any solution f of the unmodified problem (2.9)–(2.11) *satisfies*

$$\left(\sum_{p\geq 1}\|f_p(t,\cdot,\cdot)\|_{L^\infty}\right)^{2/d}\left(\sum_{p\geq 1}\iint \frac{v^2}{2}\,f_p(t,x,v)\,dx\,dv\right) < \frac{\mathcal{E}}{2} \tag{5.27}$$

for $t\in(0,T)$, where the constant $\mathcal{E}$ is the same as in Corollary 5.4*. Moreover there exists $\varepsilon_T>0$ such that if $\varepsilon<\varepsilon_T$ and if* (5.26) *is fulfilled then any solution f^ε of the regularized problem* (5.18)-(5.20) *satisfies* (5.27) *for $t\in(0,T)$.*

Proof.
The unmodified problem (2.9)–(2.11).
Let us estimate the initial total energy $\mathcal{E}_{tot}(0)$. Without loss of generality, we can assume that $\overline{\mathcal{E}}\leq \mathcal{E}/2$. Consequently Corollary 5.4 and Theorem 3.4 can be applied at $t=0$ and the initial total energy is bounded as follows:

$$\mathcal{E}_{tot}(0)\leq C\sum_{p\geq 1}\iint (1+v^2+p^2)\,f_{p,0}\,dx\,dv = CE_0.$$

For $t>0$ and $x\in\partial\omega\times(0,1)$ we have $V(t,x)=0$. Thus, with the notations of the Appendix, $\epsilon_p(t,x)=\epsilon[V_{ext}(t,x,\cdot)]$ and by (A.6) and (A.7) we have

$$\epsilon_p \leq \frac{1}{2}\pi^2p^2 + \|[V_{ext}(t)]^+\|_{L^\infty} \leq \frac{1}{2}\pi^2p^2 + \|V_{ext}(0)\|_{L^\infty} + \int_0^t \|[\partial_t V_{ext}(t)]^+\|_{L^\infty(\Omega)}\,dt.$$

Hence with (5.25) we get

$$\int_0^t \|n(s)\|_{L^1}\,\|[\partial_t V_{ext}(s)]^+\|_{L^\infty}\,ds + \alpha\sum_{p\geq 1}\int_0^t\iint_{\Sigma_-}\left(\frac{v^2}{2}+\epsilon_p\right)g_p\,d\Sigma\,ds \leq C_0(E_0+E_b),$$

where C_0 depends only on $\|V_{ext}(0)\|_{L^\infty}$ and on $\int_0^T \|[\partial_t V_{ext}(t)]^+\|_{L^\infty(\Omega)}\,dt$. Gathering this estimate and the estimate of the initial energy, (5.24) yields

$$\mathcal{E}_{tot}(t)\leq C_0(E_0+E_b).$$

Finally, since $\sum_{p\geq 1}\|f_p(t)\|_{L^\infty} \leq \max(M_0, M_b)$ (which is a consequence of the results on the linear Vlasov equation stated in Lemma 5.2), we get

$$\left(\sum_{p\geq 1}\|f_p(t,\cdot,\cdot)\|_{L^\infty}\right)^{2/d}\left(\sum_{p\geq 1}\iint \frac{v^2}{2}\, f_p(t,x,v)\,dx\,dv\right) \leq C_0(M_0+M_b)^{2/d}(E_0+E_b)$$
$$\leq C_0\overline{\mathcal{E}}.$$

The proof is concluded by choosing $C_0\overline{\mathcal{E}} < \mathcal{E}/2$.

The modified problem.
The modified problem (5.18)–(5.20) can be treated in the same way. The only modification concerns the estimate of ϵ_p^ε at the boundary, since we do not have $R^\varepsilon[V^\varepsilon] = 0$ on $\partial\omega \times (0,1)$. Nevertheless we can use the fact that this quantity is small. Let $x \in \partial\omega \times (0,1)$. Remark that

$$\epsilon_p^\varepsilon(t,x) = \epsilon_p[R^\varepsilon[V^\varepsilon] + V_{ext}](t,x) \leq \epsilon_p[R^\varepsilon[V^\varepsilon] + V_{ext}^+](t,x).$$

By (A.23) we have

$$\left|\epsilon_p[R^\varepsilon[V^\varepsilon] + V_{ext}^+] - \epsilon_p[V_{ext}^+]\right|(t,x) \leq Ce^{C_1\|V^\varepsilon\|_{L^{\infty,1}}}\ \|R^\varepsilon[V^\varepsilon](t,x,\cdot)\|_{L^1(0,1)}\,.$$

Therefore we have

$$\epsilon_p^\varepsilon(t,x) \leq \frac{1}{2}\pi^2 p^2 + \|[V_{ext}(t)]^+\|_{L^\infty} + Ce^{C_1\|V^\varepsilon\|_{L^{\infty,1}}}\ \|R^\varepsilon[V^\varepsilon](t,x,\cdot)\|_{L^1_z(0,1)}\,. \qquad (5.28)$$

We recall that by Theorem 3.4 and (5.13)

$$\|V^\varepsilon\|_{L^{\infty,1}(\Omega)}(t) \leq C\|V^\varepsilon\|_{W^{2,\frac{d+2}{d}}(\Omega)}(t) \leq C(M_0 + M_b + \mathcal{E}_{tot}^\varepsilon(t)) \qquad (5.29)$$

and that $\|R^\varepsilon[V] - V\|_{L^{\infty,1}(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0$, for any function $V \in C^0(\overline{\omega}, L^1(0,1))$ such that $V(x,z) = 0$ on $\partial\omega \times (0,1)$ (see Lemma 3.1). Since the embedding of $W^{2,\frac{d+2}{d}}(\Omega)$ in $C^0(\overline{\omega}, L^1(0,1))$ is compact, it is not difficult to show by contradiction the existence of a constant $C(\varepsilon) > 0$ such that $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$ and

$$\|R^\varepsilon[V] - V\|_{L^{\infty,1}(\Omega)} \leq C(\varepsilon)\|V\|_{W^{2,\frac{d+2}{d}}(\Omega)}.$$

Since $V^\varepsilon = 0$ on $\partial\omega \times (0,1)$, (5.29) implies that for $x \in \partial\omega$

$$\|R^\varepsilon[V^\varepsilon](t,x,\cdot)\|_{L^1_z(0,1)} \leq CC(\varepsilon)(1 + \mathcal{E}_{tot}^\varepsilon(t)).$$

Inserting this inequality and (5.29) in (5.28), then using (5.24), leads to

$$\mathcal{E}_{tot}^\varepsilon(t) \leq C_0\,(E_0 + E_b) + C(\varepsilon)\int_0^t (1 + \mathcal{E}_{tot}^\varepsilon(s))\, e^{C_1\mathcal{E}_{tot}^\varepsilon(s)}\,ds.$$

A standard perturbation argument shows that for any $\eta > 0$ there exists ε_T such that for $\varepsilon < \varepsilon_T$

$$\mathcal{E}_{tot}^\varepsilon(t) \leq C_0\,(E_0 + E_b) + \eta, \qquad t \in [0,T].$$

This leads to the estimate

$$\left(\sum_{p\geq 1}\|f_p^\varepsilon(t)\|_{L^\infty}\right)^{2/d}\left(\sum_{p\geq 1}\iint \frac{v^2}{2}\, f_p^\varepsilon(t)\,dx\,dv\right)\leq C_0\,\overline{\mathcal{E}}+\eta(M_0+M_b)^{2/d}.$$

whose right-hand side is smaller than $\mathcal{E}/2$ for η small enough. □

Proposition 5.9 *Let T be given and let $\overline{\mathcal{E}}$ and ε_T be as in Corollary* 5.8. *Assume that the initial data defined in* (2.12)-(2.13) *verify $(M_0+M_b)^{d/2}(E_0+E_b)<\overline{\mathcal{E}}$ and that $0<\varepsilon<\varepsilon_T$. Then the regularized SVP problem* (5.18)-(5.20) *admits a global solution $(V^\varepsilon,(\epsilon_p^\varepsilon,\chi_p^\varepsilon,f_p^\varepsilon)_{p\geq 1})$ on the interval* $[0,T]$.

Proof. Whenever the solution exists, since the assumptions of Corollary 5.8 are satisfied by the data, this solution satisfies the estimate (5.27) on its interval of definition.

We shall prove that –as soon as the initial data satisfy (5.27)– a solution can be constructed on an interval $[0,T_\varepsilon]$ which might depend on ε. Then, by the above remark, this solution will also verify (5.27) on the whole interval $[0,T_\varepsilon]$. The time T_ε can thus be taken as an initial time and one can extend the solution on $[T_\varepsilon,2T_\varepsilon]$ The solution will actually be defined on the whole interval $[0,T]$.

The local existence proof relies on the Schauder fixed point theorem. The fixed point mapping, that we shall denote by S_ε, is defined as follows : starting from $V\in L^{1+1/d}((0,T_\varepsilon)\times\Omega)$, we set $\epsilon_p=\epsilon_p[R^\varepsilon[V]+V_{ext}]$. The regularity of V_{ext} (assumption **(H-2)**) insures that $\epsilon_p\in L^{1+1/d}((0,T_\varepsilon),C^2(\overline{\omega}))$ which allows to construct $(f_p)_{p\geq 1}$ by solving the Vlasov equations (5.18). Then, we compute $(V^*,(\epsilon_p^*,\chi_p^*)_{p\geq 1})$ as the solution of the regularized Schrödinger-Poisson system (3.4)-(3.5) in which $\rho_p=\int f_p\,dv$. By definition, we set

$$S_\varepsilon(V)=V^*.$$

Note that the last step of the calculation is nonlinear and involves the resolution of a Schrödinger-Poisson system (we recall that in the stationary case treated in Section 4, the last step of the fixed point scheme was the resolution of a linear Poisson equation).

Well-posedness of the mapping under Condition (5.15)

Let us define the bounded, closed, convex set

$$\mathcal{K}_\tau=\{V\in L^{1+1/d}((0,\tau)\times\Omega)\ :\quad 0\leq\|V\|_{L^{1+1/d}}\leq 1\}. \tag{5.30}$$

We shall prove here that there exists $T_\varepsilon>0$ such that for $t<T_\varepsilon$ (5.15) is satisfied by the f_p's defined above, which ensures the well-posedness of the mapping S_ε, then that $S_\varepsilon(\mathcal{K}_{T_\varepsilon})\subset\mathcal{K}_{T_\varepsilon}$. To this aim, we notice that for any τ and for any $V\in\mathcal{K}_\tau$, the eigenvalues ϵ_p corresponding to the potential $R^\varepsilon[V]+V_{ext}$ satisfy

$$\|\nabla_x\epsilon_p\|_{L_t^{1+1/d}(W_x^{1,\infty})}\leq C_\varepsilon,$$

where C_ε only depends on ε and V_{ext} (and not on τ). Besides, we deduce from Lemma 5.2 that the solution f_p satisfies

$$\begin{aligned}\iint v^2 f_p(t,x,v)\,dxdv \le \; & \iint v^2 f_{p,0}\,dxdv + \alpha \int_0^t \iint_{\Sigma_-} \frac{v^2}{2}\, g_p\, d\Sigma\, ds \\ & + C(\Phi^\varepsilon(t) + \Phi^\varepsilon(t)^{d+2}),\end{aligned}$$

where

$$\begin{aligned}\Phi^\varepsilon(t) & = \max(\|g_p\|_{L^\infty}, \|f_{p,0}\|_{L^\infty})^{1/(d+2)} \int_0^t \|\nabla_x \epsilon_p(s,\cdot)\|_{L^\infty_x(\omega)}\, ds \\ & \le C_\varepsilon\, t^{1/(1+d)}.\end{aligned}$$

This ensures that

$$\begin{aligned}\left(\sum_{p\ge1} \|f_p(t)\|_{L^\infty}\right)^{2/d} \left(\sum_{p\ge1} \iint \frac{v^2}{2} f_p(t)\,dxdv\right) \le & \,(M_0+M_b)^{2/d}(E_0+E_b) \\ & + C_\varepsilon \left(t^{1/(1+d)} + t^{(d+2)/(d+1)}\right).\end{aligned}$$

Since we have assumed that $(M_0+M_b)^{2/d}(E_0+E_b) < \overline{\mathcal{E}} \le \mathcal{E}/2$, there exists $\delta_\varepsilon > 0$ such that (5.15) is fulfilled for $t \in [0,\delta_\varepsilon]$. By Corollary 5.4, the regularized Schrödinger-Poisson system (3.4)-(3.5) with $\rho_p = \int f_p\,dv$ admits a unique solution $(V^*, (\epsilon_p^*, \chi_p^*)_{p\ge1})$, which satisfies a uniform estimate

$$\|V^*\|_{L^\infty((0,\delta_\varepsilon), W^{2,\frac{d+2}{d}}(\Omega))} < C.$$

This implies that if $\tau \le \delta_\varepsilon$ the mapping S_ε is well-defined on $[0,\tau]$ and satisfies

$$\|S_\varepsilon(V)\|_{L^{1+1/d}((0,\tau)\times\Omega)} < C_0\, \tau^{d/(d+1)}, \tag{5.31}$$

where C_0 only depends on V_{ext}. Let

$$T_\varepsilon = \min\left(\delta_\varepsilon, \frac{1}{C_0^{(d+1)/d}}\right).$$

Then (5.31) shows that the set $\mathcal{K}_{T_\varepsilon}$ is stable by S_ε.

Compactness and continuity of S_ε

The compactness of S_ε is a consequence of Lemma 5.5 and its continuity is a consequence of Lemma 5.6. Indeed, consider a sequence V^n in $\mathcal{K}_{T_\varepsilon}$. We have seen that the corresponding distribution functions f_p^n satisfy (5.15) for $t \le T_\varepsilon$. Besides, by (5.10) we have after a summation

$$\sum_{p\ge1} \iint p^2\, f_p^n(t,x,v)\,dx\,dv \le E_0 + E_b.$$

Thus a Young inequality gives

$$\sum_{p\geq 1}\|f_p^n\|_{L^\infty((0,T)\times\omega\times\mathbb{R}^d)}+\sum_{p\geq 1}\|(v^2+p^2)f_p^n\|_{L^1((0,T)\times\omega\times\mathbb{R}^d)}\leq C, \tag{5.32}$$

where C is independent of n. Hence Lemma 5.5 applies and the sequence (ρ^n) is compact in $\ell^1(L^q(0,T_\varepsilon)\times\omega))$, for any $q<\frac{d+2}{d}$. Remark that $1+1/d<\frac{d+2}{d}$ and that $\max(1,d/2)<\frac{d+2}{d}$ (since $d\leq 3$). Besides by Corollary 5.4 we have $\|\rho(t,\cdot)\|_{\ell^1(L^{(d+2)/d})}<\mathcal{N}$ for $t<T_\varepsilon$. Hence, setting $q=\frac{d+2}{d}-\eta$, for $\eta>0$ small enough, q is such that

$$\|\rho(t,\cdot)\|_{\ell^1(L^q)}<\mathcal{N}\quad\text{for } t<T_\varepsilon\quad;\quad\max(1,d/2)<q<\frac{d+2}{d}\quad;\quad 1+1/d<q.$$

Therefore, up to the extraction of a subsequence, (ρ^n) converges in $\ell^1(L^q((0,T_\varepsilon)\times\omega))$ and Theorem 3.4 applies with this value of q. Consequently the sequence $S_\varepsilon(V^n)$ is a Cauchy sequence in $L^q((0,T_\varepsilon),W^{2,q}(\Omega))$ and converges in particular in $L^{1+1/d}((0,T_\varepsilon)\times\Omega)$.

Let us now prove the sequential continuity of the mapping S_ε. We assume that the sequence $V^n\in\mathcal{K}_{T_\varepsilon}$ converges towards V in the $L^{1+1/d}$ topology. From the continuity of the regularizing operator R^ε, we deduce that $R^\varepsilon[V^n]$ converges towards $R^\varepsilon[V]$ in $L^{1+1/d}((0,T)\times C^1(\overline{\Omega}))$. Let ϵ_p^n be the eigenvalues corresponding to $R^\varepsilon[V]+V_{ext}$. Lemma A.7 shows that

$$\epsilon_p^n\overset{n\to\infty}{\longrightarrow}\epsilon_p\quad\text{in } L^{1+1/d}((0,T_\varepsilon),C^1(\overline{\omega})).$$

Besides, by Lemma 5.2 the corresponding distribution functions f_p^n are bounded in $L^\infty((0,T_\varepsilon),\,L^1\cap L^\infty(\omega\times\mathbb{R}^d))$. After a diagonal extraction of subsequences, $f_p^n\rightharpoonup f_p$ in $L^\infty_{t,x,v}$ weak *. Moreover, for $t<T_\varepsilon$, f_p^n satisfies (5.32). Hence Items *(i)* and *(ii)* of Lemma 5.6 can be applied: the limit f_p is a weak solution of (5.8) with the force field $-\nabla_x\epsilon_p$ and the sequence $(\rho_p^n)_{p\geq1}$ converges to $(\rho_p)_{p\geq1}$ defined by $\rho_p=\int f_p\,dv$, in the $\ell^1(L^q(0,T_\varepsilon)\times\omega))$ topology (q is the same as above in the proof of compactness).

This limit $(f_p)_{p\geq1}$ also satisfies (5.15), thus by Corollary 5.4 and Part *(ii)* of Theorem 3.4, we have finally

$$\|S_\varepsilon(V^n)-S_\varepsilon(V)\|_{L^{1+1/d}((0,T_\varepsilon)\times\Omega)}\leq C\|(\rho^n)-(\rho)\|_{\ell^1(L^q)}\quad\overset{n\to\infty}{\longrightarrow}0.$$

Remark that the limit is unique, thus that all the terms of the initial sequence $S_\varepsilon(V^n)$ -and not only a subsequence- converge to $S_\varepsilon(V)$. □

5.4 Proof of Theorem 2.3

The proof of Theorem 2.3 is obtained by passing to the limit $\varepsilon\to0$ in the regularized problem (5.18)-(5.20). This can be done exactly like in the proof of compactness and continuity of the mapping S_ε, by exploiting the convergence properties of the operator

R^ε developed in Lemma 3.1. We shall only sketch this proof and leave the details for the reader.

By Corollary 5.8, the solution of the modified problem given by Proposition 5.9 satisfies (5.15). This has two consequences. First, the compactness Lemma 5.5 can be applied and leads the concergence as $\varepsilon \to 0$, up to an extraction, of the sequence $(\rho_p^\varepsilon)_{p\geq 1}$ in $\ell^1(L^q((0,T)\times\omega))$ (with $q<(d+2)/d$). Second, for any given time t, the quasistatic Schrödinger-Poisson part of the problem is solved in the framework of Theorem 3.4 (*i.e.* the solution is continuous with respect to (ρ^ε)).

Denote by $(V,(\epsilon_p,\chi_p)_{p\geq 1})$ the solution of the *unmodified* Schrödinger-Poisson system with the occupation factors ρ_p and by $(V^{0,\varepsilon},(\epsilon_p^{0,\varepsilon},\chi_p^{0,\varepsilon})_{p\geq 1})$ the solution of the *unmodified* Schrödinger-Poisson system with the occupation factors ρ_p^ε :

$$\begin{cases} -\dfrac{1}{2}\,\partial_{zz}\chi_p^{0,\varepsilon}+(V^{0,\varepsilon}+V_{ext})\chi_p^{0,\varepsilon}=\epsilon_p^{0,\varepsilon}\chi_p^{0,\varepsilon}\,, \\ -\Delta V^{0,\varepsilon}=\displaystyle\sum_{p\geq 1}\rho_p^\varepsilon\left|\chi_p^{0,\varepsilon}\right|^2. \end{cases}$$

By Item *(i)* of Theorem 3.4, $V^\varepsilon-V^{0,\varepsilon}$ converges to 0 in $L^\infty((0,T),W^{2,q}(\Omega))$ as $\varepsilon\to 0$. Moreover, by Item *(ii)* of the same theorem (applied to the *unmodified* S-P system), we have

$$\|V^{0,\varepsilon}-V\|_{L^q((0,T),W^{2,q}(\Omega))}\leq C\|\rho^\varepsilon-\rho\|_{\ell^1(L^q((0,T)\times\omega))}\,.$$

which implies that V^ε converges to V in $L^q((0,T),W^{2,q}(\Omega))$. Since V^ε and V belong to $W^{1,q}_\omega$, by Lemma 3.1 $R^\varepsilon[V^\varepsilon]$ converges to V in $L^q((0,T),W^{1,q}(\Omega))$. Consequently, the eigenvalues $\epsilon_p^\varepsilon:=\epsilon_p[R^\varepsilon[V^\varepsilon]+V_{ext}]$ converge in $L^q((0,T),W^{1,q}(\omega))$ as $\varepsilon\to 0$, and by Lemma 5.6 one can pass to the limit in the whole Vlasov-Schrödinger-Poisson system.

The occupation numbers $(\rho_p)_{p\geq 1}$ are in $\ell^1(L^\infty((0,T),L^{(d+2)/d}(\omega)))$. It is not difficult to show, by a similar argument to the one developed in [20], that $(\rho_p)_{p\geq 1}\in\ell^1(C([0,T],L^{(d+2)/d}(\omega)))$, which yields $V\in C([0,T],W^{2,\frac{d+2}{d}}(\Omega))$ in view of Theorem 3.4. Finally, the regularity of the ϵ_p's and the χ_p's is deduced from Lemma 4.4.

6 Comments

The stationary problem.
In the stationary problem, we have assumed that the accomodation coefficient α of the boundary condition (2.5) is different from zero, while the value $\alpha=0$ is allowed for the evolution problem. This is due to the fact that the value $\alpha=0$ corresponds to an isolated system for which the total mass has to be prescribed in the stationary case. Since the boundary operator $\mathcal{R}_p$ conserves all the functions of the energy, the techniques developed in Section 4 allow to prove the following theorem:

Theorem 6.10 *For any given positive number M and any given decreasing function $\Phi : \mathbb{R} \to \mathbb{R}^+$ such that*

$$\sum_p p \int_0^{+\infty} |u|^{d-1}\, \Phi\left(\frac{u^2}{2} + \frac{\pi^2 p^2}{2}\right) du < +\infty,$$

there exists a unique solution $(f_p, \chi_p, \epsilon_p, V)$ of (2.5)–(2.7) with $\alpha = 0$, such that

$$f_p(x,v) = \Phi\left(\frac{v^2}{2} + \epsilon_p(x) - \epsilon_F\right)$$

and

$$\sum_p \iint f_p(x,v)\, dx\, dv = M.$$

We shall only give some hints about the proof of this theorem. The first remark is that a function $\Phi(\frac{1}{2}|v|^2 + \epsilon_p(x) - \epsilon_F)$ is a solution of the Vlasov equation (2.5) with $\alpha = 0$. Consequently, the occupation numbers $\rho_p = \int f_p\, dv$ are decreasing with respect to p, locally in x. This ensures, in view of Proposition 3.8, the existence and uniqueness of a solution V of the Schrödinger-Poisson problem (3.1)-(3.2). The overall problem can be now rewritten as a minimization problem under the constraint $\sum_p \iint f_p(x,v)\, dx\, dv = M$ which can be solved uniquely (ϵ_F being the Lagrange multiplier associated to this constraint). We shall not develop this point here and refer to [22, 23, 28].

The time-dependent problem.
The time-dependent problem is solved only for small data. This is not due to a failure of an *a priori* estimate (the energy estimate holds without the smallness hypothesis). The reason is that we were not able to prove that the Schrödinger-Poisson system (3.1)-(3.2) has a unique solution for large data, neither were we able to select a solution continuously depending on the data (for large data). This fact implied a lack of time compactness which is necessary for the construction of a solution for the Vlasov-Schrödinger-Poisson system. On the other hand Proposition 3.8 insures the existence and uniqueness of the solution of (3.1)-(3.2) when the occupation numbers ρ_p are pointwise (in t and x) decreasing with respect to p. This decay property is trivially preserved during time evolution in the so-called electrical quantum limit, where for $p \geq 2$ we have $f_p^0 = 0$ and $g_p = 0$. In this case, only the first subband is occupied and we have the following theorem:

Theorem 6.11 *Under the additional hypothesis $f_p^0 = 0$ and $g_p = 0$ for $p \geq 2$, the results of Theorem 2.3 hold true with $\overline{\mathcal{E}} = +\infty$.*

The Vlasov-Schrödinger system (without Poisson coupling) was obtained by the authors in [8] as the partial semiclassical limit of the Schrödinger equation. The coupling with Poisson equation will be tackled in the future in the framework of the electrical quantum limit for which the Vlasov-Schrödinger-Poisson system has better properties than the most general case.

Collisions.
We have assumed all along this paper that the transport in the x direction is ballistic so that the Vlasov equation has to be solved. Existence theorems similar to Theorems 2.2 and 2.3 can be proven when collisions are taken into account. The Vlasov equations are replaced by Boltzmann type equations

$$\partial_t f_p + v \cdot \nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = Q(f)_p$$

where $f = (f_n)_{n \in \mathbb{N}^*}$ and Q is a matrix collision operator which models intersubband transition (extra-diagonal terms) and intrasubband transitions (diagonal terms).

For the linear, diagonal, elastic collision operator, the existence Theorems 2.2 and 2.3 apply without any change. Namely, take

$$Q(f)_p = Q_p(f_p) = \int \alpha_p(t, x, v, v') \, (f(v') - f(v)) \, \delta(|v|^2 - |v'|^2) \, dv'$$

and assume that α_p is nonnegative, symmetric (with respect to v and v') and satisfies

$$\int \alpha_p(t, x, v, v') \delta(|v|^2 - |v'|^2) \, dv' \in L^\infty$$

then the results of Sections 2–5 hold true when the Vlasov equation

$$\partial_t f_p + v \cdot \nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = 0$$

is replaced by the Boltzmann equation

$$\partial_t f_p + v \cdot \nabla_x f_p - \nabla_x \epsilon_p \cdot \nabla_v f_p = Q(f)_p.$$

The reason for this is that Q_p conserves any function of the energy, that it conserves the L^∞ norm and that any isotropic (in velocity) function belongs to the kernel of Q_p.

In a forthcoming paper [10], a Drift-Diffusion-Schrödinger-Poisson system, obtained thanks to a diffusion limit of a Boltzmann-Schrödinger-Poisson system is studied.

Appendix

A Spectral properties of Sturm-Liouville operators

In this appendix, we present some basic properties satisfied by the eigenvalues and eigenfunctions of the one-dimensional Schrödinger operator. Most of these properties can be found in [32, 40, 44]. Let U be a real-valued function in $L^2(0,1)$ and let $H[U]$ be the Dirichlet Schrödinger operator

$$H[U] = -\frac{1}{2}\frac{d^2}{dz^2} + U(z) \tag{A.1}$$

defined on the domain

$$D(H[U]) = H^2(0,1) \cap H_0^1(0,1). \tag{A.2}$$

The operator $H[U]$ is a selfadjoint operator on $L^2(0,1)$, bounded from below and with compact resolvent. There exists a strictly increasing sequence $(\epsilon_p[U])_{p\geq 1}$ of real numbers tending to $+\infty$ and an orthonormal basis of $L^2(0,1)$ $(\chi_p[U](z))_{p\geq 1}$ such that

$$\begin{cases} -\dfrac{1}{2}\dfrac{d^2}{dz^2}\chi_p + U\chi_p = \epsilon_p\chi_p\,, \\ \chi_p \in H_0^1(0,1), \qquad \displaystyle\int_0^1 \chi_p\,\chi_q\,dz = \delta_{pq}\,. \end{cases} \tag{A.3}$$

The ϵ_p's are the eigenvalues of $H[U]$ while the χ_p's are the corresponding eigenfunctions. It is readily seen from (A.3) that

$$\epsilon_p = \frac{1}{2}\int_0^1 \left|\frac{d}{dz}\chi_p(z)\right|^2 dz + \int_0^1 U(z)\,|\chi_p(z)|^2\,dz. \tag{A.4}$$

Besides, the eigenvalues are simple (this is why the sequence (ϵ_p) is strictly increasing) and for $U = 0$ they are given by

$$\epsilon_p[0] = \frac{1}{2}\pi^2 p^2 \quad ; \quad \chi_p[0](z) = \sqrt{2}\sin(\pi p z).$$

In the general case, the eigenvalues ϵ_p are given by the Min-Max formula [32]

$$\epsilon_p[U] = \max_{\dim E_p = p-1} \min_{\substack{\phi\in E_p^\perp\cap H_0^1(0,1) \\ \|\phi\|_{L^2}=1}} \left(\int_0^1 \frac{1}{2}\left|\frac{d\phi}{dz}(z)\right|^2 dz + \int_0^1 U(z)\,|\phi(z)|^2\,dz\right). \tag{A.5}$$

An immediate consequence of the Min-Max formula is

$$\text{if} \qquad U \geq V \quad \text{a.e. on } (0,1) \qquad \text{then} \qquad \forall p \in \mathbb{N}^* \quad \epsilon_p(U) \geq \epsilon_p(V). \tag{A.6}$$

Another consequence is the

Lemma A.1
(i) Let U and V be two real-valued functions in $L^2(0,1)$ such that $U-V \in L^\infty(0,1)$. Then the corresponding eigenvalues verify

$$|\epsilon_p[U] - \epsilon_p[V]| \leq \|U - V\|_{L^\infty(0,1)}. \tag{A.7}$$

The particular case $V = 0$ gives

$$\left|\epsilon_p[U] - \frac{1}{2}\pi^2\, p^2\right| \leq \|U\|_{L^\infty(0,1)}. \tag{A.8}$$

(ii) Let U be a function in $L^\infty(0,1)$. Then there exists a constant C independent of U and of p such that

$$\|\chi_p[U]\|_{H^1(0,1)} \leq C\left(p + \|U\|^{1/2}_{L^\infty(0,1)}\right), \tag{A.9}$$

$$\forall r \in [1,+\infty] \qquad \|\chi_p[U]\|_{L^r(0,1)} \leq C\left(1 + p^{1/2-1/r} + \|U\|^{1/4-1/(2r)}_{L^\infty(0,1)}\right). \tag{A.10}$$

Proof. The fact that ϵ_p is Lipschitz with respect to U in the L^∞ topology can be proved directly using (A.5). Let us now prove Item *(ii)*. Since $\|\chi_p\|_{L^2_z} = 1$, Estimate (A.10) in the case $r \leq 2$ is trivial. In the case $r \geq 2$, it is a simple consequence of (A.9) and of the Gagliardo-Nirenberg estimate

$$\|f\|_{L^r} \leq C\|f\|^{1/2+1/r}_{L^2}\|f\|^{1/2-1/r}_{H^1} \qquad \forall f \in H^1(0,1).$$

To prove (A.9), we simply remark that (A.4) and (A.8) imply $\int_0^1 |\frac{d}{dz}\chi_p(z)|^2\,dz \leq \frac{1}{2}\pi^2p^2 + 2\|U\|_{L^\infty_z}$. ◻

For Section 3, it is interesting to consider the Schrödinger operator with $L^1(0,1)$ potentials. In [44], the spectral properties of $H[U]$ are studied in the framework of Sturm-Liouville operators, without invoking the abstract theory of selfadjoint operators. The results of Chapter 2 of [44] were obtained for $L^2(0,1)$ but the proofs are still valid for $L^1(0,1)$ potentials. We state these results in the following two lemmas:

Lemma A.2 *Let $U \in L^1(0,1)$. Then the eigenvalue problem* (A.3) *admits a unique solution $(\epsilon_p, \chi_p)_{p\geq 1}$. The sequence $(\epsilon_p)_{p\geq 1}$ is bounded from below and strictly increasing to $+\infty$. The sequence $(\chi_p)_{p\geq 1}$ is an orthonormal basis of $L^2(0,1)$.*

Lemma A.3 *Let $U \in L^1(0,1)$. Then there exists a positive constant C^1_U depending only on $\|U\|_{L^1(0,1)}$ such that*

$$\left|\epsilon_p[U] - \frac{1}{2}\pi^2p^2\right| \leq C^1_U \tag{A.11}$$

$$\left\|\chi_p[U] - \sqrt{2}\sin(\pi pz)\right\|_{L^\infty(0,1)} + \frac{1}{p}\left\|\frac{d}{dz}\chi_p[U] - \sqrt{2}\pi\, p\cos(\pi pz)\right\|_{L^\infty(0,1)} \leq \frac{C^1_U}{p}. \tag{A.12}$$

Moreover the constant C_U^1 can be chosen such that

$$C_U^1 \leq C_1 \exp\left(C_2 \|U\|_{L^1(0,1)}\right),$$

where the constants C_1 and C_2 are independant of U.

Remark A.4 One of the consequences of Lemmas A.2 and A.3 is that the mappings

$$\epsilon_p : L^1(0,1) \to \mathbb{R} \quad ; \quad \chi_p : L^1(0,1) \to C([0,1])$$

are weakly continuous. In particular, (A.6) and Item *(i)* of Lemma A.1 are still valid when the space $L^2(0,1)$ is replaced by $L^1(0,1)$.

With the bounds of ϵ_p and of $\|\chi_p[U]\|_{W^{1,\infty}(0,1)}$ given by Lemma A.3, we deduce the

Lemma A.5 *Let $U \in L^\alpha(0,1)$ with $\alpha > 1$. Then there exists a positive constant δ_U^α depending only on $\|U\|_{L^\alpha(0,1)}$, such that*

$$\forall (p,q) \in (\mathbb{N}^*)^2 \qquad |\epsilon_p[U] - \epsilon_q[U]| \geq \delta_U^\alpha\, |p-q|^2. \tag{A.13}$$

Proof. If $p = q$, this inequality is obvious. Let us first prove that there exists a constant δ_U^α, independent of p, q and depending only on $\|U\|_{L^\alpha(0,1)}$, such that

$$\min_{p \neq q} |\epsilon_p[U] - \epsilon_q[U]| \geq \delta_U^\alpha. \tag{A.14}$$

If (A.14) was false, it would be possible to find a sequence (U^n) weakly converging to U in $L^\alpha(0,1)$, thus in $L^1(0,1)$, and a sequence (p^n) of integers such that $\epsilon_{p^n+1}[U^n] - \epsilon_{p^n}[U^n]$ converges to zero as n tends to $+\infty$. The asymptotic behaviour of the ϵ_p's deduced from (A.11) implies that the sequence (p^n) is bounded. Therefore, up to an extraction, it is stationary : $p^n = p$. Besides, from Remark A.4 we deduce that $\epsilon_p[U^n]$ and $\epsilon_{p+1}[U^n]$ converge to $\epsilon_p[U]$ and $\epsilon_{p+1}[U]$. Hence $\epsilon_p[U] = \epsilon_{p+1}[U]$, which is in contradiction with Lemma A.2 (the eigenvalues are strictly increasing).

Let us now prove (A.13). From (A.11) we have

$$\frac{\pi^2}{2} p^2 - C_U^1 \leq \epsilon_p \leq \frac{\pi^2}{2} p^2 + C_U^1,$$

which gives, for any (p,q):

$$|\epsilon_p - \epsilon_q| \geq \frac{\pi^2}{2}|q-p|^2 + \pi^2|q-p| - 2C_U^1\,.$$

Hence if $|q-p| \geq C_U^1$ then $|\epsilon_p - \epsilon_q| \geq \frac{\pi^2}{2}|q-p|^2$. From this inequality and (A.14) we can deduce easily (A.13) (up to a change of δ_U^α). □

The following lemma contains additional information on the U dependence of eigenfunctions and eigenvalues when the potential depends on a parameter.

Lemma A.6 *Let $V = V(\lambda, z) \in L^\infty_{\mathrm{loc}}(\lambda, L^1_z(0,1))$ where λ is a real parameter (typically $\lambda = t$ or $\lambda = x_i$). Let us denote $\epsilon_p(\lambda)$ instead of $\epsilon_p[V(\lambda,\cdot)]$ and analogously for $\chi_p(\lambda)$. Assume that $\partial_\lambda V \in L^1_{\mathrm{loc}}(\lambda, L^1_z(0,1))$.*
(i) Then $\partial_\lambda \epsilon_p \in L^1_{\mathrm{loc}}$ and we have

$$\partial_\lambda \epsilon_p(\lambda) = \int_0^1 |\chi_p(\lambda,z)|^2 \,\partial_\lambda V(\lambda,z)\,dz, \tag{A.15}$$

$$|\partial_\lambda \epsilon_p(\lambda)| \le C_V^1 \,\|\partial_\lambda V(\lambda,\cdot)\|_{L^1_z(0,1)}. \tag{A.16}$$

(ii) If $V \in L^\infty_{\mathrm{loc}}(\lambda, L^\alpha_z(0,1))$ with $\alpha > 1$ then $\partial_\lambda \chi_p \in L^1_{\mathrm{loc}}(\lambda, L^\infty_z(0,1))$ and we have

$$\partial_\lambda \chi_p(\lambda,z) = \sum_{q\neq p} \frac{\left(\displaystyle\int_0^1 \chi_p(\lambda,z')\,\chi_q(\lambda,z')\,\partial_\lambda V(\lambda,z')\,dz'\right)}{\epsilon_p(\lambda)-\epsilon_q(\lambda)}\,\chi_q(\lambda,z)\,, \tag{A.17}$$

$$\|\partial_\lambda \chi_p(\lambda,\cdot)\|_{L^\infty_z(0,1)} \le C_V^\alpha \,\|\partial_\lambda V(\lambda,\cdot)\|_{L^1_z(0,1)}. \tag{A.18}$$

(iii) If $V \in L^\infty_{\mathrm{loc}}(\lambda, L^\alpha_z(0,1))$ with $\alpha > 1$ and $\partial_\lambda V \in L^1_{\mathrm{loc}}(\lambda, L^\gamma_z(0,1))$ with $1 < \gamma \le 2$ then $\partial_{\lambda z}\chi_p \in L^1_{\mathrm{loc}}(\lambda, L^\infty_z(0,1))$ and

$$\|\partial_{\lambda z}\chi_p(\lambda,\cdot)\|_{L^\infty_z(0,1)} \le C_V^\alpha \,p\,\|\partial_\lambda V(\lambda,\cdot)\|_{L^\gamma_z(0,1)}. \tag{A.19}$$

(iv) If $V \in L^\infty_{\mathrm{loc}}(\lambda, L^\alpha_z(0,1))$ with $\alpha > 1$ and $\partial_{\lambda\lambda} V \in L^1_{\mathrm{loc}}(\lambda, L^1_z(0,1))$ then $\partial_{\lambda\lambda}\epsilon_p \in L^1_{\mathrm{loc}}$ and $\partial_{\lambda\lambda}\chi_p \in L^1_{\mathrm{loc}}(\lambda, L^\infty_z(0,1))$ and we have the pointwise (in λ) estimate

$$|\partial_{\lambda\lambda}\epsilon_p(\lambda)| + \|\partial_{\lambda\lambda}\chi_p(\lambda,\cdot)\|_{L^\infty_z(0,1)} \le C_V^\alpha \left(\|\partial_\lambda V(\lambda,\cdot)\|^2_{L^1_z(0,1)} + \|\partial_{\lambda\lambda} V(\lambda,\cdot)\|_{L^1_z(0,1)}\right). \tag{A.20}$$

In the whole Lemma, the estimates are pointwise (in λ) and the constant C_V^α depends only on $\|V(\lambda,\cdot)\|_{L^\alpha_z(0,1)}$ and not on the index p.

Proof. We first prove this lemma when the potential V is regular (say C^∞). In this case $\epsilon_p(\lambda)$ and $\chi_p(\lambda,z)$ are regular (see for instance [32]).

Since the χ_p form an orthonormal basis of $L^2(0,1)$, $\partial_\lambda\chi_p$ writes $\sum_{q\ge1} a_{p,q}\,\chi_q$. From $\|\chi_p(\lambda,\cdot)\|_{L^2_z(0,1)} = 1$ we deduce $a_{p,p} = 0$. A differentiation of (A.3) with respect to λ gives

$$\sum_{q\neq p} a_{p,q}\,(\epsilon_q - \epsilon_p)\;\chi_q = (\partial_\lambda\epsilon_p - \partial_\lambda U)\chi_p. \tag{A.21}$$

Multiplying (A.21) by χ_p and integrating with respect to z leads to (A.15). Analogously, (A.17) is obtained by multiplying (A.21) by χ_q, for $q \neq p$, and integrating with respect to z.

The inequalities (A.16) and (A.18) are obtained by straightforward estimates of the right-hand sides of (A.15) and (A.17) (in which we use the estimates (A.12) and (A.13)).

To prove Item *(iii)* we differentiate (A.17) with respect to z :

$$\partial_{\lambda z}\chi_p(\lambda, z) = \sum_{q\neq p} \frac{\left(\int_0^1 \chi_p(\lambda, z')\,\chi_q(\lambda, z')\,\partial_\lambda V(\lambda, z')\,dz'\right)}{\epsilon_p(\lambda) - \epsilon_q(\lambda)}\,\partial_z\chi_q(\lambda, z)\,.$$

Denote

$$b^\chi_{p,q} = \int_0^1 \chi_p(\lambda, z')\,\chi_q(\lambda, z')\,\partial_\lambda V(\lambda, z')\,dz'.$$

$$b^0_{p,q} = \int_0^1 \chi_p(\lambda, z')\,\sqrt{2}\sin(\pi q z')\,\partial_\lambda V(\lambda, z')\,dz',$$

By (A.12) we have

$$|b^\chi_{p,q} - b^0_{p,q}| \leq \frac{C^1_U}{q}\|\partial_\lambda V(\lambda, \cdot)\|_{L^1_z(0,1)} \leq \frac{C^1_U}{q}\|\partial_\lambda V(\lambda, \cdot)\|_{L^\gamma_z(0,1)}.$$

Moreover $b^0_{p,q}$ can be estimated thanks to the Hausdorff-Young inequality [16]: let $f \in L^s(0,1)$ with $1 \leq s \leq 2$ and let s' be the conjugate of s. Denoting the Fourier coefficients of f by

$$\hat{f}_q = \int_0^1 \sqrt{2}\sin(\pi q z)\,f(z)\,dz,$$

the series $(\hat{f}_q)_{q\geq 1}$ belongs to $\ell^{s'}$ and the following inequality holds:

$$\|(\hat{f}_q)_{q\geq 1}\|_{\ell^{s'}} \leq \|f\|_{L^s(0,1)}.$$

Here, applying the Hausdorff-Young inequality to $b^0_{p,q}$ yields the estimate

$$\left(\sum_q |b^0_{p,q}|^{\gamma'}\right)^{1/\gamma'} \leq C^1_U\|\partial_\lambda V(\lambda, \cdot)\|_{L^\gamma_z(0,1)}.$$

Using (A.13) and (A.12), we deduce

$$\|\partial_{\lambda z}\chi_p(\lambda, \cdot)\|_{L^\infty_z(0,1)} \leq C^\alpha_U\left(\sum_{q\neq p}\frac{\|\partial_\lambda V(\lambda, \cdot)\|_{L^\gamma_z(0,1)}}{|q-p|^2} + \sum_{q\neq p}\frac{q\,|b^0_{p,q}|}{|q-p|^2}\right)$$

the right-hand side being bounded by $C\|\partial_\lambda V(\lambda, \cdot)\|_{L^\gamma_z(0,1)}$ (since $\gamma > 1$).

For Item *(iv)*, we first differentiate (A.15) with respect to λ and get

$$\partial_{\lambda\lambda}\epsilon_p = \int_0^1 |\chi_p|^2\,\partial_{\lambda\lambda}V\,dz + 2\int_0^1 \chi_p\,\partial_\lambda\chi_p\,\partial_\lambda V\,dz.$$

The estimate on $\partial_{\lambda\lambda}\epsilon_p$ in (A.20) is deduced by using (A.18) and Lemma A.3. Next, differentiating twice (A.3) with respect to λ leads to

$$\begin{aligned}\partial_{\lambda\lambda}\chi_p = \quad & \sum_{q\neq p} \frac{\left(\displaystyle\int_0^1 \chi_p\,\chi_q\,\partial_{\lambda\lambda}V\,dz' + 2\int_0^1 (\partial_\lambda V - \partial_\lambda \epsilon_p)\,\chi_q\,\partial_\lambda\chi_p\,dz'\right)}{\epsilon_p - \epsilon_q}\,\chi_q \\ & -\chi_p \int_0^1 |\partial_\lambda \chi_p|^2\,dz'\end{aligned} \tag{A.22}$$

(the coefficient of χ_p in this decomposition is obtained separately by differentiating twice the equality $\int |\chi_p|^2\,dz = 1$). The convergence of the series appearing in the right-hand side is proven by using (A.16), (A.18) and Lemma A.5. This leads to (A.20).

When the potential is not regular, we regularize it by a convolution, obtain the estimates for the regularized version and then pass to the limit in the regularization, thanks to Remark A.4. For any p the series of the right-hand sides of (A.17) and (A.22) converge uniformly. Indeed, the coefficients of χ_q behave like $1/q^2$ for q large.

□

We finish this section by the following lemma:

Lemma A.7
(i) Let U and V be two real valued functions in $L^1_z(0,1)$. There exists a constant $C^1_{U,V}$ only depending on $\|U\|_{L^1_z(0,1)}$, $\|V\|_{L^1_z(0,1)}$ (not on the index p) such that the following inequality holds:

$$|\epsilon_p[U] - \epsilon_p[V]| \leq C^1_{U,V}\|U - V\|_{L^1_z(0,1)} \tag{A.23}$$

(ii) Let U and V be two real valued functions in $L^\alpha_z(0,1)$ with $\alpha > 1$. There exists a constant $C^\alpha_{U,V}$ only depending on $\|U\|_{L^\alpha_z(0,1)}$, $\|V\|_{L^\alpha_z(0,1)}$ (not on the index p) such that the following inequalities hold:

$$\|\chi_p[U] - \chi_p[V]\|_{L^\infty_z} \leq C^\alpha_{U,V}\,\|U - V\|_{L^1_z(0,1)}. \tag{A.24}$$

$$\|\chi_p[U] - \chi_p[V]\|_{H^1_z(0,1)} \leq p\,C^\alpha_{U,V}\,\|U - V\|_{L^1_z(0,1)}. \tag{A.25}$$

(iii) Let $U = U(\lambda, z)$ and $V = V(\lambda, z)$ be in $L^\infty_{\mathrm{loc}}(\lambda, L^\alpha_z(0,1))$ where λ is a real parameter and $\alpha > 1$. Assume that $\partial_\lambda U$ and $\partial_\lambda V$ belong to $L^1(\lambda, L^1_z(0,1))$. Then we have

$$\begin{aligned}|\partial_\lambda\epsilon_p[U(\lambda,\cdot)] - \partial_\lambda\epsilon_p[V(\lambda,\cdot)]| \leq C^\alpha_{U,V}\,\big(&\|U - V\|_{L^1_z(0,1)}\,\|\partial_\lambda U\|_{L^1_z(0,1)} \\ &+\|\partial_\lambda U - \partial_\lambda V\|_{L^1_z(0,1)}\big)\,,\end{aligned} \tag{A.26}$$

where $C^\alpha_{U,V}$ depends only on $\|U\|_{L^\alpha_z(0,1)}$, $\|V\|_{L^\alpha_z(0,1)}$.

Proof. We shall see that (A.23) and (A.24) are consequences of Lemma A.6. To this aim, let $\lambda \in [0,1]$ and define $W(\lambda, z) = U + \lambda(V - U)$. Denoting $\epsilon_p(\lambda) = \epsilon_p[W(\lambda,\cdot)]$ and $\chi_p(\lambda) = \chi_p[W(\lambda,\cdot)]$, we have

$$\epsilon_p[V] - \epsilon_p[U] = \epsilon_p(1) - \epsilon_p(0) = \int_0^1 \partial_\lambda \epsilon_p(\lambda)\, d\lambda,$$

$$\chi_p[V] - \chi_p[U] = \chi_p(1) - \chi_p(0) = \int_0^1 \partial_\lambda \chi_p(\lambda)\, d\lambda.$$

We then apply (A.16) and (A.18) in which we have noticed that $\partial_\lambda W = V - U$, and obtain (A.23) and (A.24).

In order to prove (A.25), we start from

$$-\frac{1}{2}\frac{d^2}{dz^2}(\chi_p - \tilde{\chi}_p) + V\,(\chi_p - \tilde{\chi}_p) = \epsilon_p\,(\chi_p - \tilde{\chi}_p) + (U - V)\,\tilde{\chi}_p + (\epsilon_p - \tilde{\epsilon}_p)\,\tilde{\chi}_p\,,$$

where χ_p and ϵ_p are quantities related to V and $\tilde{\chi}_p$ and $\tilde{\epsilon}_p$ are related to U. Multiplying by $\chi_p - \tilde{\chi}_p$ and integrating with respect to z we get

$$\begin{aligned}
\frac{1}{2}\int_0^1 \left|\frac{d}{dz}(\chi_p - \tilde{\chi}_p)\right|^2 dz \;&= \epsilon_p\,\|\chi_p - \tilde{\chi}_p\|^2_{L^2_z} - \int_0^1 V\,|\chi_p - \tilde{\chi}_p|^2\,dz \\
&\quad + \int_0^1 (U - V)\,\tilde{\chi}_p\,(\chi_p - \tilde{\chi}_p)\,dz + \int_0^1 (\epsilon_p - \tilde{\epsilon}_p)\,\tilde{\chi}_p\,(\chi_p - \tilde{\chi}_p)\,dz \\
&\le \epsilon_p\,\|\chi_p - \tilde{\chi}_p\|^2_{L^2_z(0,1)} + \|V\|_{L^1_z(0,1)}\,\|\chi_p - \tilde{\chi}_p\|^2_{L^\infty_z(0,1)} \\
&\quad + \left(\|U - V\|_{L^1} + |\epsilon_p - \tilde{\epsilon}_p|\right)\|\tilde{\chi}_p\|_{L^\infty_z(0,1)}\,\|\chi_p - \tilde{\chi}_p\|_{L^\infty_z(0,1)}.
\end{aligned}$$

Using (A.11), (A.12), (A.23) and (A.24), we finally deduce that

$$\int_0^1 \left|\frac{d}{dz}(\chi_p - \tilde{\chi}_p)\right|^2 dz \;\le p^2\, C^\alpha_{U,V}\,\|U - V\|^2_{L^1_z(0,1)}$$

and (A.25) is proved.

To prove Item *(iii)*, we start from (A.15):

$$\begin{aligned}
\partial_\lambda \epsilon_p[U(\lambda,\cdot)] - \partial_\lambda \epsilon_p[V(\lambda,\cdot)] = &\int_0^1 \left(\chi_p[U]^2 - \chi_p[V]^2\right)\partial_\lambda U\,dz \\
&+ \int_0^1 \chi_p[V]^2\,\left(\partial_\lambda U - \partial_\lambda V\right)\,dz,
\end{aligned}$$

then apply (A.12) and (A.24). □

B Anisotropic Sobolev embeddings

The aim of this appendix is to prove the following Sobolev embedding lemma (recall that $\Omega = \omega \times (0,1)$ is a bounded domain of $\mathbb{R}^{d+1}$):

Lemma B.1 *Let $s > 1$. The following Sobolev embeddings hold true:*
(i) $\underline{1 < s < d}$. Let $1 \le p \le +\infty$, $s \le q \le +\infty$ be such that

$$\frac{d}{p(d+1)} + \frac{1}{q(d+1)} \ge \frac{1}{s} - \frac{1}{d+1}, \tag{B.1}$$

Then $W^{1,s}(\Omega) \subset L^{p,q}(\Omega)$. If (B.1) *holds strictly then the embedding is compact.*
(ii) $\underline{1 < s = d}$. Let $1 \le p < +\infty$ and $1 \le q \le +\infty$. Then $W^{1,s}(\Omega)$ is compactly embedded in $L^{p,q}(\Omega)$ if

$$\frac{d}{p} + \frac{1}{q} > \frac{1}{d}. \tag{B.2}$$

If $p \le q$ and if (B.2) *is an equality then $W^{1,s}(\Omega)$ is continuously embedded in $L^{p,q}(\Omega)$.*
(iii) $\underline{d < s < d+1}$. If $1 \le p, q \le +\infty$ satisfy (B.1) *then $W^{1,s}(\Omega) \subset L^{p,q}(\Omega)$. The embedding is compact if* (B.1) *holds strictly.*
(iv) $\underline{d/2 < s}$. Then $W^{2,s}(\Omega) \subset C^0(\overline{\omega}, L^s(0,1))$ and the embedding is compact.

Proof. The proof of *(i), (ii), (iii)* relies on the following three arguments.
Argument 1: Gagliardo Nirenberg inequality. Let $r \ge s$ be given. for almost every $x \in \omega$, the Gagliardo-Nirenberg inequality [14] gives

$$\|f(x,\cdot)\|_{L^\infty_z(0,1)} \le C \|f(x,\cdot)\|^{1-\theta}_{L^r_z(0,1)} \|f(x,\cdot)\|^{\theta}_{W^{1,s}_z(0,1)}$$

with $\theta = s/(rs+s-r)$. Let $p \ge 1$ be defined by $\frac{p(1-\theta)}{r} + \frac{p\theta}{s} = 1$, *i.e.* $p = 1 + r(1-1/s)$. By raising the above inequality to the power p and integrating with respect to x we get

$$\|f\|^p_{L^{p,\infty}(\Omega)} \le C \int_\omega \|f(x,\cdot)\|^{p(1-\theta)}_{L^r_z(0,1)} \|f(x,\cdot)\|^{p\theta}_{W^{1,s}_z(0,1)} \, dx.$$

The Hölder inequality leads to

$$\|f\|_{L^{p,\infty}(\Omega)} \le C \|f\|^{1-\theta}_{L^r(\Omega)} \|f\|^{\theta}_{W^{1,s}(\Omega)}.$$

Let us now choose $r \le r_* = \frac{(d+1)s}{d+1-s}$ so that $W^{1,s}(\Omega) \subset L^r(\Omega)$. The corresponding p satisfies $p \le p_* = \frac{ds}{d+1-s}$ (*i.e.* (B.1) is satisfied with $q = +\infty$). By noticing that the embedding $W^{1,s}(\Omega) \subset L^r(\Omega)$ is compact for $r < r_*$, it is readily seen that the embedding $W^{1,s}(\Omega) \subset L^{p,\infty}(\Omega)$ for $p < p_*$ is compact.
Argument 2: Sobolev embedding in ω. We remark that if $f \in W^{1,s}(\Omega)$, then $g(x) = \|f(x,\cdot)\|_{L^s_z}$ is in $W^{1,s}(\omega)$ and

$$\|g\|_{W^{1,s}(\omega)} \le \|f\|_{W^{1,s}(\Omega)}$$

Therefore,

- If $s < d$, $g \in L^p$ for $p \leq p_{**} = \frac{sd}{d-s}$, which leads to $W^{1,s}(\Omega) \subset L^{p_{**},s}$.
- If $s = d$, $g \in L^p$ for $p < +\infty$, which leads to $W^{1,s}(\Omega) \subset L^{p,s}(\Omega)\ \forall p < +\infty$.
- If $s > d$, $g \in C^0(\overline{\omega})$ which leads to $W^{1,s}(\Omega) \subset C^0(\overline{\omega}, L^s(0,1)) \subset L^{\infty,s}(\Omega)$.

Argument 3: Interpolation. It is readily seen that if

$$\frac{\theta}{p_1} + \frac{1-\theta}{p_2} = \frac{1}{p_3} \quad ; \quad \frac{\theta}{q_1} + \frac{1-\theta}{q_2} = \frac{1}{q_3}$$

for some $\theta \in [0,1]$, then

$$\|f\|_{L^{p_3,q_3}} \leq \|f\|^{\theta}_{L^{p_1,q_1}} \|f\|^{1-\theta}_{L^{p_2,q_2}}.$$

Let us consider the case $s < d$. From the first step, we have $W^{1,s} \subset L^{p,q}$ for all $p \leq p_*$ and $q \leq +\infty$. The second step insures $W^{1,s} \subset L^{p,q}$ for $q \leq s$ and $p \leq p_{**}$. From the interpolation inequality, we deduce that $W^{1,s} \subset L^{p,q}$ for all (p,q) such that $(1/p, 1/q) \in \mathcal{C}$ where $\mathcal{C}$ is the convex envelope of $[1/p_*, 1] \times [0,1] \cup [1/p_{**}, 1] \times [1/s, 1]$. This is the desired result. Indeed, the equation of the segment $(1/p, 1/q) \in [(1/p_*, 0), (1/p_{**}, 1/s)]$ is (B.1) with an equality. The compactness embedding is a consequence of the same interpolation argument and of the compactness result obtained in the step 1 for $p < p_*$, $q = \infty$. The cases $s \geq d$ are treated analogously and the details are left to the reader. The three cases are summarized in Figure 1.

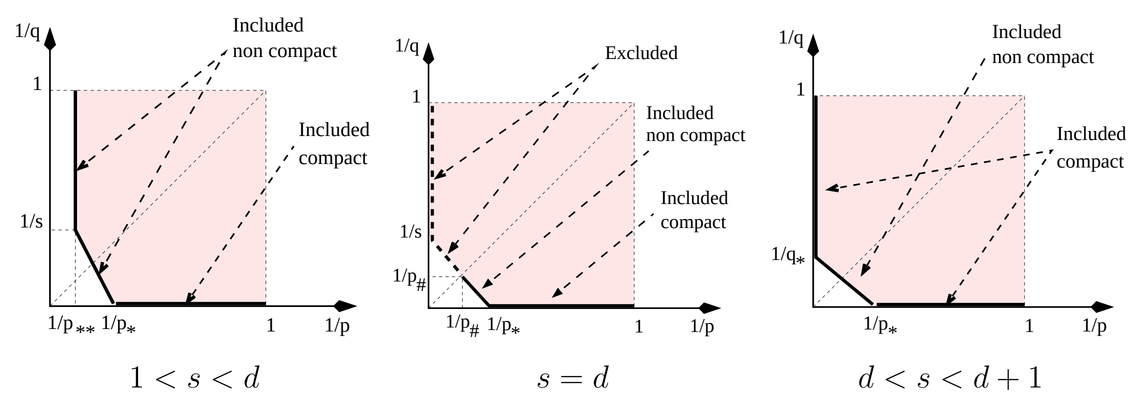

Figure 1: Sobolev embeddings of $W^{1,s}$ in $L^{p,q}$

Two limiting cases. If $s = d$ then we have the standard Sobolev embedding in dimension $d+1$: $W^{1,s}(\Omega) \subset L^p(\Omega)$ where $p = p_\# = \frac{(d+1)s}{d+1-s}$. Then by interpolation we recover the continuous embedding in $L^{p,q}(\Omega)$ for all the segment $(1/p, 1/q) \in [(1/p_\#, 1/p_\#), (1/p_*, 0)]$.

In the case $d < s < d+1$ it remains to prove that the embedding in $L^{\infty,q}$ is compact for $q < q_* = \frac{s}{d+1-s}$. We apply a Gagliardo-Nirenberg inequality as in the first step, but starting with the x variable: with a suitable $\tilde{\theta}$ we have

$$\|f(\cdot, z)\|_{L^\infty_x(\omega)} \leq C \|f(\cdot, z)\|^{1-\tilde{\theta}}_{L^r_x(\omega)} \|f(\cdot, z)\|^{\tilde{\theta}}_{W^{1,s}_x(\omega)}.$$

Then with $q \geq 1$ defined by $\frac{q(1-\tilde{\theta})}{r} + \frac{q\tilde{\theta}}{s} = 1$ and a Hölder inequality we get

$$\|f\|_{L^{\infty,q}(\Omega)} \leq \left(\int_0^1 \|f(\cdot,z)\|^q_{L^\infty(\omega)} dz\right)^{1/q} \leq C\|f\|^{1-\tilde{\theta}}_{L^r(\Omega)} \|f\|^{\tilde{\theta}}_{W^{1,s}(\Omega)}.$$

We conclude by choosing $r < r_*$, which gives $q < q_*$.

To prove *(iv)*, we proceed as follows : since $W^{2,s}(\omega) \subset C^{0,\alpha}(\overline{\omega})$ for some $\alpha > 0$, there exists a constant $C > 0$ such that for any given $(x, x') \in \omega \times \omega$, we have

$$\int_0^1 |V(x,z) - V(x',z)|^s \, dz \leq C|x-x'|^{\alpha s} \int_0^1 \|V(\cdot,z)\|^s_{W^{2,s}(\omega)} \, dz \leq C|x-x'|^{\alpha s} \|V\|^s_{W^{2,s}(\Omega)}.$$

Therefore $W^{2,s}(\Omega) \subset C^{0,\alpha}(\overline{\omega}, L^s(0,1)) \subset C^0(\overline{\omega}, L^s(0,1))$. The compactness of the embedding is a consequence of the compactness of $W^{2,s}(\Omega) \subset W^{2-\varepsilon,s}(\Omega)$ ($\varepsilon > 0$ is chosen small enough).

□

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