

# Global existence of classical solutions for a Vlasov-Schrödinger-Poisson system

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## Abstract

Global existence and uniqueness of a classical solution of the two dimensional Vlasov equation coupled to the Schrödinger-Poisson system is proven. The two dimensional driving forces appearing in the Vlasov equation are deduced from the electrostatic potential in the Born-Oppenheimer approximation and take into account the quantum behaviour in the third direction. The electrostatic potential is computed by solving a three dimensional Poisson equation. The existence and uniqueness of the solution is proven by a fixed point argument on the Vlasov equation. It relies on the use of an a priori energy estimate, and on the resolution of the Schrödinger-Poisson system by convex minimization.

## 1 Introduction

A coupled Vlasov-Schrödinger-Poisson model was presented and analyzed in [3]. This system, based on the so-called *subbands*, models the transport of charges in a nanostructure under a partial confinement. The lengthscale of the confinement direction is of the same order of magnitude as the de Broglie length of the charges, therefore they have a quantum behaviour in this direction, referred to as the *transverse* direction. In the other directions, that we shall call the *parallel* directions, the lengthscale is larger and the charges are transported semiclassically. The model presented here takes advantage of these different lengthscales by coupling quantum and kinetic effects.

The two first authors have studied this system in [3] in a bounded domain and showed the existence of weak solutions. The aim of this paper is to carry on the analysis of this system in the whole space case and to show the existence and uniqueness of a strong solution.

More precisely, let us denote by  $x \in \mathbb{R}^2$  the parallel variable and by  $z \in \mathbb{R}$  the transverse one. The problem consists in finding, for  $t \in (0, T)$ ,  $x \in \mathbb{R}^2$ ,  $v \in \mathbb{R}^2$ , a distribution function  $f(t, x, v)$  solving the Vlasov equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \epsilon \cdot \nabla_v f = 0, \\ f(0, x, v) = f_0(x, v), \end{cases} \quad (1.1)$$

where  $\epsilon(t, x)$  is the first eigenvalue of the following quasistatic Schrödinger problem in the transverse variable  $z \in \mathbb{R}$

$$\begin{cases} -\frac{1}{2} \partial_z^2 \chi + (V + V_{ext})\chi = \epsilon \chi, \\ \int_{\mathbb{R}} |\chi(t, x, z)|^2 dz = 1. \end{cases} \quad (1.2)$$

The selfconsistent potential  $V(t, x, z)$  satisfies the Poisson equation

$$V = \frac{1}{4\pi r} * n \quad ; \quad n = \left( \int_{\mathbb{R}^2} f dv \right) |\chi|^2, \quad (1.3)$$

where we have denoted  $r = \sqrt{|x|^2 + z^2}$ ; the Cauchy datum  $f_0$  is given as well as the external confining potential  $V_{ext}$ , which satisfies the following growth condition:

**Assumption 1.1** *The external potential  $V_{ext} = V_{ext}(z)$  belongs to  $C^2(\mathbb{R})$  and satisfies  $V_{ext}(z) \rightarrow +\infty$  as  $|z| \rightarrow +\infty$ .*

The quasistatic Schrödinger equation (1.2) is an eigenvalue problem in the one-dimensional variable  $z$ , the variables  $t$  and  $x$  appearing as parameters, and  $\epsilon$  can also be defined by

$$\epsilon = \min_{\substack{\phi \in H^1(\mathbb{R}) \\ \|\phi\|_{L^2(\mathbb{R})} = 1}} \left( \frac{1}{2} \int_{\mathbb{R}} \left| \frac{d}{dz} \phi(z) \right|^2 dz + \int_{\mathbb{R}} (V + V_{ext}) |\phi(z)|^2 dz \right), \quad (1.4)$$

while  $\chi$  (unique up to a sign) realizes this minimum. More generally, the eigenvalue problem (1.2) defines the *subbands* of the system, which are the eigenspaces of the partial Hamiltonian  $-\frac{1}{2} \partial_z^2 + V + V_{ext}$  (its spectrum is discrete thanks to Assumption 1.1). By considering only the first subband in this system, we study here a particular case of the quantum-kinetic subband problem [3], usually referred to as the *electrical quantum limit* in the physics literature [7, 6]. The population of carriers is described by the transversal wavefunction  $\chi(t, x, z)$ , and the two-dimensional distribution function  $f(t, x, v)$ , which is a solution of the Vlasov equation (1.1) driven by the effective, Born-Oppenheimer, electric field  $-\nabla_x \epsilon$  (see [18] and references therein for related problems in quantum chemistry). Collisions are not taken into account in this model and the charges interact only through the electrostatic potential  $V(t, x, z)$  computed by solving the Poisson equation (1.3). We assume that the initial data satisfies the

**Assumption 1.2** *The initial data  $f_0$  belongs to  $C^1(\mathbb{R}^4) \cap W^{1,\infty}(\mathbb{R}^4)$  and satisfies for any  $(x, v) \in \mathbb{R}^4$*

$$0 \leq f_0(x, v) \leq C(1 + |x|)^{-2\gamma}(1 + |v|)^{-2\gamma}, \quad (1.5)$$

for some constant  $\gamma > 3$ . Moreover  $\nabla_x f_0 \in L^1(\mathbb{R}^4)$ ,  $\nabla_v f_0 \in L^1(\mathbb{R}^4)$  and we have

$$|\nabla_x f_0(x, v)| + |\nabla_v f_0(x, v)| \leq C(1 + |v|)^{-\gamma}. \quad (1.6)$$

The main result of this paper is the

**Theorem 1.3** *Under Assumptions 1.1 and 1.2, the system (1.1)–(1.3) admits globally in time a unique classical solution.*

To prove this theorem, we shall take advantage of the similarities between (1.1)–(1.3) and the standard Vlasov-Poisson system. Indeed, the system can be seen as a time-dependent Vlasov equation (1.1) coupled with a quasistatic Schrödinger-Poisson system (1.2), (1.3) instead of the Poisson equation. The Vlasov equation in our problem is two-dimensional. Its coupling with the Poisson equation (in dimension 2) was studied by Ukai and Okabe [19] and by Wollman [20].

Our problem is more complicated than the two-dimensional Vlasov-Poisson system (but simpler than the three dimensional one, for which one can refer to [15, 17, 2, 9, 10, 1, 11, 21] and the reviews in [5, 8]). Indeed, the linear Poisson equation is replaced by the nonlinear Schrödinger-Poisson system. Therefore, an important part of this work will concern the resolution of this quasistatic problem. Afterwards, in order to analyze the coupling of this problem with the two-dimensional Vlasov equation, we shall adapt the proofs developed by Ukai and Okabe in [19] for the two-dimensional Vlasov-Poisson system. However, due to the three-dimensional nature of the quasistatic Schrödinger-Poisson system, this coupling will only give in a first step a local-in-time solution. Nevertheless, we can go beyond this difficulty and construct a global-in-time solution by using a natural energy estimate for the whole system, which relies on the repulsive nature of the Coulomb interaction.

This paper is organized as follows. Next section is devoted to the analysis of the Schrödinger-Poisson system for a given  $f$ . We show that this system is well posed. Then in Section 3 we give the natural *a priori* estimates for the whole system. In Section 4 we construct the global classical solution. Finally an appendix gives some useful technical results for the Schrödinger, Vlasov and Poisson equations.

Before going further, let us define some functional spaces that will be used all along this paper. Like in [19], we introduce for  $k \in \mathbb{N}^*$ ,  $\alpha \in [0, 1)$ , and for a subset  $\Omega$  in  $\mathbb{R}^m$ , the space  $B^{k+\alpha}(\Omega)$  of  $k$  times continuously differentiable functions on  $\Omega$  such that the derivatives until the order  $k$  are bounded and Hölder continuous with exponent  $\alpha$  (if  $\alpha > 0$ ). The set  $B^{\alpha_1, k+\alpha_2}([0, T] \times \mathbb{R}^2)$  denotes the space of functions  $f(t, x)$  which are  $k$  times continuously differentiable with respect to the  $x$  variable and such that their derivatives are bounded and Hölder continuous with respect to the  $(t, x)$  variables, with the exponents  $\alpha_1$  in the  $t$  variable and  $\alpha_2$  in the  $x$  variable. For any function  $f(z)$ , we sometimes use the notation

$$\langle f \rangle := \int_{\mathbb{R}} f(z) dz.$$

For any  $1 \leq p \leq +\infty$ ,  $1 \leq q \leq +\infty$  we set

$$L_x^p L_z^q = \left\{ f \in L_{loc}^1(\mathbb{R}^3) \quad \text{such that} \quad \left( \int_{\mathbb{R}^2} \|f(x, \cdot)\|_{L^q(\mathbb{R})}^p \right)^{1/p} < +\infty \right\}$$

(with an obvious generalization for  $p = +\infty$ ). The norm on  $L^p(\mathbb{R}^d)$ ,  $d = 1, 2$  or  $3$ , will be denoted by  $\|\cdot\|_p$ , while the norm on  $L_x^p L_z^q$  will be denoted by  $\|\cdot\|_{p,q}$ .

We will need the following Sobolev embedding result:

**Lemma 1.4** *The space*

$$\mathcal{H} = \{f \in L^6(\mathbb{R}^3) : \partial_z f \in L^2(\mathbb{R}^3)\}$$

*is continuously embedded in  $L_x^4 L_z^\infty$  and there exists a constant  $C > 0$  such that*

$$\forall f \in \mathcal{H}, \quad \|f\|_{4,\infty} \leq C \|f\|_6^{3/4} \|\partial_z f\|_2^{1/4}. \quad (1.7)$$

**Proof.** For almost every  $x \in \mathbb{R}^2$ , the Gagliardo-Nirenberg inequality in dimension 1 gives

$$\|f(x, \cdot)\|_\infty \leq C \|f(x, \cdot)\|_6^{3/4} \|\partial_z f(x, \cdot)\|_2^{1/4}.$$

By raising the above inequality to the power 4 and integrating with respect to  $x$  we get

$$\|f\|_{4,\infty}^4 \leq C \int_{\mathbb{R}^2} \|f(x, \cdot)\|_6^3 \|\partial_z f(x, \cdot)\|_2 dx,$$

which leads to (1.7) after applying a Hölder inequality.  $\square$

Finally we shall denote by

$$\mathcal{K} = \{f \in L^6(\mathbb{R}^3) : \nabla_{x,z} f \in L^2(\mathbb{R}^3)^3\} \quad (1.8)$$

which is a Hilbert space when endowed with the scalar product  $\int_{\mathbb{R}^3} \nabla_{x,z} u \cdot \nabla_{x,z} v dx dz$  thanks to the Sobolev inequality

$$\forall V \in \mathcal{K} \quad \|V\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla_{x,z} V\|_{L^2(\mathbb{R}^3)}. \quad (1.9)$$

## 2 The Schrödinger-Poisson system

This section is devoted to the study of a subproblem of the whole coupled system (1.1)–(1.3), namely, the quasistatic Schrödinger-Poisson system (1.2), (1.3). We shall assume along this section that the surface density  $\rho(x)$  is given (except in Lemma 2.5,  $\rho$  will be independent of time for simplicity, since  $t$  appears only as a parameter). We consider the following equation:

$$\begin{cases} -\frac{1}{2} \partial_z^2 \chi + (V + V_{ext}) \chi = \epsilon \chi, \\ \int_{\mathbb{R}} |\chi(t, x, z)|^2 dz = 1, \end{cases} \quad (2.1)$$

where  $\epsilon$  is specified to be the first eigenvalue of this problem, coupled with the Poisson equation

$$V = \frac{1}{4\pi r} * (\rho |\chi|^2). \quad (2.2)$$

More precisely, for any function  $V \in L^\infty(\mathbb{R})$ , we introduce the operator  $H[V] = -\frac{1}{2}\frac{d^2}{dz^2} + (V + V_{ext})$  on  $L^2(\mathbb{R})$  with the domain

$$D(H) = \{\psi \in H^2(\mathbb{R}) : V_{ext} \psi \in L^2(\mathbb{R})\}.$$

Some spectral properties of this operator are listed in Appendix A. Thanks to Assumption 1.1, this operator has a discrete spectrum and the eigenvalues are simple and will be denoted by  $\epsilon_p[V]$  for  $p \geq 1$  (we shall denote by  $\chi_p[V]$  the corresponding eigenfunction). A triplet  $(V, \epsilon, \chi)$  is a solution of (2.1), (2.2), if and only if  $\epsilon(x) = \epsilon_1[V(x, \cdot)]$  and  $\chi(x, z) = \chi_1[V(x, \cdot)](z)$  and  $V$  is a solution of the nonlinear-nonlocal elliptic equation

$$V = \frac{1}{4\pi r} * (\rho |\chi_1[V]|^2). \quad (2.3)$$

**Theorem 2.1** *Let  $\rho \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$  such that  $\rho \geq 0$ . The system (2.1), (2.2) admits a unique solution such that  $\epsilon \in W^{1,\infty}(\mathbb{R}^2)$ . Moreover, the following estimates hold:*

$$\|\nabla_x \epsilon\|_\infty \leq C(\|\rho\|_1^{1/3} \|\rho\|_\infty^{2/3} + \|\rho\|_1^{1/2} \|\rho\|_\infty^{5/6}), \quad (2.4)$$

where  $C$  is independent of  $\rho$ , and

$$\forall x, x' \quad |\nabla_x \epsilon(x) - \nabla_x \epsilon(x')| \leq C_\rho \zeta(|x - x'|), \quad (2.5)$$

where  $C_\rho$  depends only on the  $L^1$  and  $L^\infty$  norms of  $\rho$  and the function  $\zeta$  is defined by  $\zeta(y) = y(1 - \ln y)$  for  $0 \leq y \leq 1$  and  $\zeta(y) = y$  for  $y > 1$ . Furthermore, for two data  $\rho$  and  $\tilde{\rho}$  in  $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ , the corresponding solutions satisfy

$$\|\epsilon - \tilde{\epsilon}\|_{W^{1,\infty}(\mathbb{R}^2)} \leq C_{\rho, \tilde{\rho}} (\|\rho - \tilde{\rho}\|_1 + \|\rho - \tilde{\rho}\|_\infty), \quad (2.6)$$

where  $C_{\rho, \tilde{\rho}}$  depends only on the  $L^1$  and  $L^\infty$  norms of  $\rho$  and  $\tilde{\rho}$ .

## 2.1 Existence and uniqueness of the solution

Following [12, 13, 14, 3], we shall solve (2.1), (2.2) by a variational method. To this aim, for  $\rho$  given as a nonnegative function in  $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ , we introduce the functional

$$J(V) = J_0(V) + J_\rho(V); \quad J_0(V) = \frac{1}{2} \iint_{\mathbb{R}^3} |\nabla_{x,z} V|^2 dx dz; \quad J_\rho(V) = - \int_{\mathbb{R}^2} \epsilon_1[V] \rho dx.$$

We shall see that this functional is convex and Gâteaux differentiable on  $\mathcal{K}$  defined by (1.8) (and not in  $H^1$  as in [12, 13, 14, 3]) and that its minimizer satisfies the quasistatic Schrödinger-Poisson system.

**Lemma 2.2** *Let  $\rho \in L^{4/3}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$  such that  $\rho \geq 0$ . The functional  $J$  is coercive, continuous and strictly convex on  $\mathcal{K}$ . It admits a single minimizer on  $\mathcal{K}$  denoted by  $V$ , which is the unique weak solution in  $\mathcal{K}$  of (2.1), (2.2).*

**Proof.** The first part of the functional, defined by

$$J_0(V) = \frac{1}{2} \iint_{\mathbb{R}^3} |\nabla_{x,z} V|^2 dx dz$$

is clearly continuous and strictly convex on  $\mathcal{K}$ . Now we have to ensure that the functional  $J_\rho$  is well defined. To this aim, we use the inequality (A.4): for any  $U, V$  in  $\mathcal{K}$ , we have, pointwise in  $x$ ,

$$|\epsilon_1[V(x, \cdot)] - \epsilon_1[U(x, \cdot)]| \leq \|V(x, \cdot) - U(x, \cdot)\|_\infty.$$

Then we take the  $L^4(\mathbb{R}^2)$  norm of this difference, apply Lemma 1.4 and (1.9) and finally get

$$\|\epsilon_1[V] - \epsilon_1[U]\|_4 \leq \|V - U\|_{4,\infty} \leq C \|\nabla_{x,z} V - \nabla_{x,z} U\|_{L^2(\mathbb{R}^3)}. \quad (2.7)$$

Taking  $U = 0$  in this expression, we get

$$\|\epsilon_1[V] - \epsilon_1[0]\|_4 \leq C \|\nabla_{x,z} V\|_{L^2(\mathbb{R}^3)},$$

which implies that  $\epsilon_1[V] \in L^4(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ . Since  $\rho \in L^{4/3}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ , the functional  $J_\rho$  is well-defined on  $\mathcal{K}$ . Furthermore, (2.7) proves that the function  $U \mapsto \epsilon_1[U]$  is Lipschitz continuous from  $\mathcal{K}$  to  $L^4(\mathbb{R}^2)$ . Therefore the functional  $J_\rho$  is Lipschitz continuous on  $\mathcal{K}$ :

$$|J_\rho(V) - J_\rho(U)| \leq C(\rho) \|\nabla_{x,z} V - \nabla_{x,z} U\|_{L^2(\mathbb{R}^3)}.$$

Moreover, by (1.4) and pointwise in  $x$ ,  $\epsilon_1[V]$  is defined as the minimum with respect to  $\phi$  of the function

$$E(\phi, V) = \frac{1}{2} \int_{\mathbb{R}} \left| \frac{d}{dz} \phi \right|^2 dz + \int_{\mathbb{R}} (V + V_{ext}) |\phi|^2 dz$$

which is affine in  $V$ . Thus  $\epsilon_1[V]$  is concave with respect to  $V$  and  $J_\rho$  is convex. We have proved that the functional  $J$  is continuous and convex on  $\mathcal{K}$ . Its coercivity is obvious, thanks to (2.7), which gives

$$J(V) \geq \frac{1}{2} \|\nabla_{x,z} V\|_{L^2}^2 - C \|\rho\|_{4/3} \|\nabla_{x,z} V\|_{L^2} - J_\rho(0).$$

Let us now check that the critical points of  $J$  are the weak solutions of (2.1), (2.2). Since  $\rho \in L^{4/3}(\mathbb{R}^2)$ , we deduce from Lemma A.1 that  $J_\rho$  is differentiable on  $L_x^4 L_z^\infty$ . Lemma 1.4 insures the embedding  $\mathcal{K} \rightarrow L_x^4 L_z^\infty$  which in turn yields the differentiability of  $J_\rho$  on  $\mathcal{K}$ . Moreover, for  $V, W$  in  $\mathcal{K}$ , the derivative of  $J_\rho$  at  $V$  in the direction  $W$  is given by

$$d_V J_\rho(W) = - \iint_{\mathbb{R}^3} \rho(x) |\chi_1[V(x, \cdot)]|^2(z) W(x, z) dx dz.$$

Since we also have

$$d_V J_0(W) = \iint_{\mathbb{R}^3} \nabla_{x,z} V \cdot \nabla_{x,z} W \, dx \, dz,$$

the proof is complete.  $\square$

Let us now give some further integrability and regularity properties of the solution of (2.1), (2.2).

**Lemma 2.3** *Let  $\rho \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$  such that  $\rho \geq 0$ . Then the solution  $V, \chi, \epsilon$  of (2.1), (2.2) satisfies the following estimates:*

$$\|V\|_\infty + \|\epsilon - \epsilon_1[0]\|_\infty \leq C \|\rho\|_1^{1/2} \|\rho\|_\infty^{1/2}, \quad (2.8)$$

$$\|\nabla_{x,z} V\|_\infty + \|\nabla_x \epsilon\|_\infty \leq C (\|\rho\|_1^{1/3} \|\rho\|_\infty^{2/3} + \|\rho\|_1^{1/2} \|\rho\|_\infty^{5/6}), \quad (2.9)$$

where  $C$  is independent of  $\rho$ , and

$$\|\chi\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C(\rho). \quad (2.10)$$

where  $C$  is independent of  $\rho$  and the constant  $C(\rho)$  only depends on  $\|\rho\|_1$  and  $\|\rho\|_\infty$ .

**Proof.** Setting  $n(x, z) = \rho(x) |\chi(x, z)|^2$ , the normalization of the eigenvector  $\chi(x, \cdot)$  in  $L^2(\mathbb{R})$  implies that

$$\|n\|_1 = \|\rho\|_1 \quad ; \quad \|n\|_{\infty,1} = \|\rho\|_\infty.$$

Besides, we have

$$|V(x, z)| = \left| \frac{1}{4\pi r} * n \right|.$$

By (B.8) with  $p = \infty$ , we obtain the estimate of  $V$  in (2.8); the  $L^\infty$  estimate of  $\epsilon$  in (2.8) is deduced from (A.4). Next, (A.6) and (2.8) yield

$$\|\chi\|_\infty \leq C(1 + \|\rho\|_1^{1/8} \|\rho\|_\infty^{1/8}),$$

thus we have

$$\|n\|_\infty \leq C(\|\rho\|_\infty + \|\rho\|_1^{1/4} \|\rho\|_\infty^{5/4});$$

(B.6) and (A.8) give (2.9). Finally (2.10) is obtained by using (2.8), (2.9), (A.6) and (A.10).  $\square$

**Lemma 2.4** *Let  $\rho \in L^1 \cap L^\infty(\mathbb{R}^2)$  such that  $\rho \geq 0$ . Then the solution of (2.1), (2.2) satisfies the estimate*

$$\|\nabla_x \epsilon\|_\infty \leq C(\rho) \|\rho\|_\infty^{1/2}, \quad (2.11)$$

where  $C(\rho)$  is a generic constant which only depends on  $\|\rho\|_1$  and  $\|\rho\|_2$ .

**Proof.** This lemma will be proved in two steps. First, for any  $p \in [1, 2]$ , we interpolate the  $L_x^p L_z^1$  norm of  $n = \rho |\chi|^2$  between its  $L^1$  norm and its  $L_x^2 L_z^1$  norms and obtain

$$\|n\|_{p,1} \leq C(\rho).$$

Next, for any  $q \in (2, \infty)$ , the inequality (B.7) with  $p = 2q/(q+2) \in (1, 2)$  leads to

$$\forall q \in (2, \infty) \quad \|V\|_{q,\infty} \leq C(\rho)$$

and by (A.6) we get

$$\forall q \in (4, \infty) \quad \|\chi^2(x, \cdot)\|_{L_z^\infty} \in L_x^\infty + L_x^q,$$

with norms bounded by  $C(\rho)$ . Since  $\rho \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ , it is readily seen that

$$\forall p \in [1, 2) \quad \|n\|_{p,\infty} \leq C(\rho).$$

Then we apply (B.5) with, for instance,  $p = 7/4$  and get the following estimates independent of  $\|\rho\|_\infty$ :

$$\|V\|_\infty + \|\chi\|_\infty \leq C(\rho). \quad (2.12)$$

where we have also used (A.6). Now, the second step consists in writing

$$\nabla_x \epsilon = \langle \nabla_x V |\chi|^2 \rangle. \quad (2.13)$$

Applying (B.8) with  $p = \infty$  and using (2.12), we obtain

$$\|\nabla_x V\|_{\infty,1} \leq C \|\rho\|_\infty^{1/2} \|\rho\|_1^{1/2}.$$

Therefore, (2.13) and the  $L^\infty(\mathbb{R}^3)$  estimate of  $\chi$  from (2.12) lead to (2.11).  $\square$

## 2.2 Proof of Theorem 2.1

Existence and uniqueness of the solution is dealt with in Lemma 2.2. Inequality (2.4) was given in (2.9). Let us now prove (2.5) and (2.6). We begin by (2.6). For simplicity, we shall denote by  $C$  instead of  $C_{\rho, \tilde{\rho}}$  a positive constant which only depends on the  $L^\infty$  and  $L^1$  norms of  $\rho$  and  $\tilde{\rho}$ . We shall also use the short notation  $\|\cdot\|$  for  $\|\cdot\|_1 + \|\cdot\|_\infty$ . In order to prove (2.6), it is enough to prove that

$$\|V - \tilde{V}\|_{W^{1,\infty}} \leq C \|\rho - \tilde{\rho}\|_1 + \|\rho - \tilde{\rho}\|_\infty \quad (2.14)$$

and then to apply (A.4) and (A.12). The proof of (2.14) will be done in two steps. First we show the result for the  $\mathcal{K}$  norm (*i.e.* for  $\|\nabla_{x,z} V - \nabla_{x,z} \tilde{V}\|_{L^2}$ ), and then we use this intermediate result to prove the Lipschitz dependence in the  $W^{1,\infty}$  norm.

**First step: estimates in  $\mathcal{K}$ .** By using the Euler-Lagrange equations for  $V$  and  $\tilde{V}$ , we obtain

$$\begin{aligned} \iint_{\mathbb{R}^3} |\nabla_{x,z}(V - \tilde{V})|^2 dx dz &= \int_{\mathbb{R}^2} (\rho - \tilde{\rho}) \langle |\tilde{\chi}|^2 (V - \tilde{V}) \rangle dx \\ &\quad + \int_{\mathbb{R}^2} \rho \langle (|\chi|^2 - |\tilde{\chi}|^2) (V - \tilde{V}) \rangle dx. \end{aligned} \quad (2.15)$$

The first term of the right-hand side can be estimated, by using (A.1), as

$$\int_{\mathbb{R}^2} (\rho - \tilde{\rho}) \langle |\tilde{\chi}|^2 (V - \tilde{V}) \rangle dx \leq C \|\rho - \tilde{\rho}\|_{6/5} \|V - \tilde{V}\|_6.$$

The second term is nonpositive. Indeed, the function  $\hat{\epsilon}(\lambda) := \epsilon[(V + \lambda(\tilde{V} - V))(x, \cdot)]$  is concave with respect to the real variable  $\lambda$  which implies that  $\frac{d\hat{\epsilon}}{d\lambda}(0) \geq \frac{d\hat{\epsilon}}{d\lambda}(1)$ . This leads, in view of Lemma A.1, to the inequality

$$\forall x \in \mathbb{R}^2, \quad \langle (|\chi|^2 - |\tilde{\chi}|^2)(V - \tilde{V}) \rangle \leq 0.$$

A more general version of such an inequality can be found in [12, 13, 14]. Consequently, we have

$$\|\nabla_{x,z}V - \nabla_{x,z}\tilde{V}\|_2^2 \leq C \|\rho - \tilde{\rho}\|_{6/5} \|V - \tilde{V}\|_6$$

which implies, in view of (1.9),

$$\|V - \tilde{V}\|_6 \leq C \|\nabla_{x,z}V - \nabla_{x,z}\tilde{V}\|_2 \leq C \|\rho - \tilde{\rho}\|_{6/5}. \quad (2.16)$$

By applying (1.7), we have

$$\|V - \tilde{V}\|_{4,\infty} \leq C \|\rho - \tilde{\rho}\|_{6/5}. \quad (2.17)$$

**Step 2 : bootstrapping.** We have

$$V - \tilde{V} = \delta V_1 + \delta V_2 \quad ; \quad \delta V_i = \frac{1}{4\pi r} * u_i, \quad (2.18)$$

where

$$u_1 = (\rho - \tilde{\rho})|\tilde{\chi}|^2 \quad ; \quad u_2 = \rho(|\chi|^2 - |\tilde{\chi}|^2). \quad (2.19)$$

For  $\delta V_1$ , the estimate is immediate, since (B.5), (B.6) imply

$$\|\delta V_1\|_{W^{1,\infty}} \leq C \|\rho - \tilde{\rho}\| \|\chi\|_\infty^2 \leq C \|\rho - \tilde{\rho}\|. \quad (2.20)$$

For  $\delta V_2$ , more care is needed. Indeed, (A.11) leads to

$$\|\delta V_2\|_{W^{1,\infty}} \leq C \|\chi - \tilde{\chi}\|_\infty \leq C \|V - \tilde{V}\|_\infty. \quad (2.21)$$

The proof will be complete, once the inequality

$$\|V - \tilde{V}\|_\infty \leq C \|\rho - \tilde{\rho}\| \quad (2.22)$$

is proved. To do so, we deduce from (B.5) that

$$\|V - \tilde{V}\|_\infty \leq \|\delta V_1\|_\infty + \|\delta V_2\|_\infty \leq C\|\rho - \tilde{\rho}\| + C\|u_2\|_2^{2/3}\|u_2\|_1^{1/3}.$$

Besides, we deduce from (A.1) and (A.11) that

$$\|u_2\|_1 \leq \|\rho\|_{4/3} \|\chi + \tilde{\chi}\|_{\infty,1} \|\chi - \tilde{\chi}\|_{4,\infty} \leq C\|V - \tilde{V}\|_{4,\infty}.$$

Similarly

$$\|u_2\|_2 \leq \|\rho\|_4 \|\chi + \tilde{\chi}\|_{\infty,2} \|\chi - \tilde{\chi}\|_{4,\infty} \leq C\|V - \tilde{V}\|_{4,\infty}.$$

From these inequalities and (2.17), we deduce that

$$\|V - \tilde{V}\|_\infty \leq C\|\rho - \tilde{\rho}\|$$

which shows in view of (2.20) and (2.21) that

$$\|V - \tilde{V}\|_{W^{1,\infty}} \leq C\|\rho - \tilde{\rho}\| \quad (2.23)$$

and concludes the proof of (2.6).

Let us now prove (2.5). This is done in the spirit of [19] ([Lemma 4.1]). We first deduce from (A.11) that

$$\begin{aligned} |\chi(x', z) - \chi(x, z)| &\leq C \sup_{z \in \mathbb{R}} |V(x, z) - V(x', z)| \\ &\leq C|x - x'| \|V\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C|x - x'|. \end{aligned}$$

Therefore

$$\begin{aligned} |\nabla_x \epsilon(x') - \langle \nabla_x V(x', \cdot) |\chi(x, \cdot)|^2 \rangle| &= |\langle \nabla_x V(x', \cdot) (|\chi(x', \cdot)|^2 - |\chi(x, \cdot)|^2) \rangle| \\ &\leq C|x - x'| \|\nabla_x V\|_\infty \|\chi\|_{\infty,1} \leq C|x - x'|. \end{aligned}$$

Consequently, we have

$$|\nabla_x \epsilon(x) - \nabla_x \epsilon(x')| \leq \langle |\nabla_x V(x', \cdot) - \nabla_x V(x, \cdot)| |\chi(x, \cdot)|^2 \rangle + C|x - x'|.$$

Let us now estimate the integral in the right-hand side:

$$\begin{aligned} &\langle |\nabla_x V(x', \cdot) - \nabla_x V(x, \cdot)| |\chi(x, \cdot)|^2 \rangle \\ &\leq \iiint_{\mathbb{R}^4} G(x - x'', x' - x'', z - z'') |\chi(x, z)|^2 |\chi(x'', z'')|^2 |\rho(x'')| dx'' dz'' dz \end{aligned}$$

with

$$G(u, v, w) = \left| \frac{u}{(|u|^2 + |w|^2)^{3/2}} - \frac{v}{(|v|^2 + |w|^2)^{3/2}} \right|.$$

As in [19], we cut the integral into two parts, setting  $d = |x - x'|$  and  $\Sigma = \{x'' \in \mathbb{R}^2 : |x - x''| \leq 2d\}$ . If  $x'' \in \Sigma$ , we have  $|x' - x''| \leq 3d$  and we write

$$G(x - x'', x' - x'', z - z'') \leq \frac{|x - x''|}{(|x - x''|^2 + (z - z'')^2)^{3/2}} + \frac{|x' - x''|}{(|x' - x''|^2 + (z - z'')^2)^{3/2}}.$$

Then we remark that

$$\int_{\mathbb{R}} \frac{|u|}{(|u|^2 + w^2)^{3/2}} dw = \frac{C}{|u|}. \quad (2.24)$$

Thus by applying this equality with  $w = z - z''$  we obtain

$$\begin{aligned} \iint_{\mathbb{R}^2} \int_{\Sigma} &\leq C \|\chi\|_{\infty} \int_{\mathbb{R}} \int_{\Sigma} \left( \frac{1}{|x - x''|} + \frac{1}{|x' - x''|} \right) |\chi(x, z)|^2 |\rho(x'')| dz dx'' \\ &= C \|\chi\|_{\infty} \int_{\Sigma} \left( \frac{1}{|x - x''|} + \frac{1}{|x' - x''|} \right) |\rho(x'')| dx'' \\ &\leq Cd \|\rho\|_{\infty}. \end{aligned}$$

On the other hand, for  $x'' \in \mathbb{R}^2 \setminus \Sigma$  we have  $|x' - x''| \geq |x - x''| - |x - x'| \geq \frac{1}{2}|x - x''|$ . Besides, straightforward calculations show that for any  $(u, v, w)$  we have

$$|G(u, v, w)| \leq \max \left( \frac{|u - v|}{(|u|^2 + w^2)^{3/2}}, \frac{|u - v|}{(|v|^2 + w^2)^{3/2}} \right).$$

Therefore, by using again (2.24), we have

$$\begin{aligned} \iint_{\mathbb{R}^2} \int_{\mathbb{R}^2 \setminus \Sigma} &\leq \iint_{\mathbb{R}^2} \int_{\mathbb{R}^2 \setminus \Sigma} \frac{|x - x'| |\chi(x, z)|^2 |\chi(x'', z'')|^2}{\left(\frac{1}{4}|x - x''|^2 + |z - z''|^2\right)^{3/2}} |\rho(x'')| dx'' dz'' dz \\ &\leq Cd \|\chi\|_{\infty} \int_{|x - x''| > 2d} \frac{|\rho(x'')|}{|x - x''|^2} dx'', \\ &\leq Cd \left( \int_{2d < |x - x''| < R} + \int_{|x - x''| \geq R} \right) \leq Cd \left( \|\rho\|_{\infty} \ln \frac{R}{d} + \frac{1}{R^2} \|\rho\|_1 \right), \end{aligned}$$

for any  $R > 2d$ . This ends the proof of (2.5).  $\square$

We end this section with the following result, which deals with a time-dependent function  $\rho(t, x)$ :

**Lemma 2.5** *Let  $\rho = \rho(t, x)$  belong to  $B^{\alpha}([0, T] \times \mathbb{R}^2)$ , where  $0 < \alpha < 1$ , and satisfy*

$$0 \leq \rho(t, x) \leq C(1 + |x|)^{-\gamma}, \quad \gamma > 3. \quad (2.25)$$

*Then the solution  $\epsilon$  of (2.1), (2.2) belongs to  $B^{\lambda\alpha, 2+\lambda\alpha}(\mathbb{R}^2)$ , for any  $\lambda \in (0, 1 - 3/\gamma)$ .*

**Proof.** This result stems from the elliptic regularity results and decay estimates for the potential  $V$ . Indeed, since  $\chi$  decays exponentially as  $|z|$  becomes large (see(A.1)), we immediately deduce from (2.25) that the density  $n = \rho|\chi|^2$  satisfies

$$|n(t, x, z)| \leq C(1 + |x| + |z|)^{-\gamma}. \quad (2.26)$$

Besides, since  $V$  satisfies (2.8) and (2.9), it is Lipschitz continuous with respect to the  $(x, z)$  variable. Moreover, by (2.23) and the assumption on  $\rho$ ,  $V$  is Hölder continuous with exponent  $\alpha$  with respect to the  $t$  variable. Inequality (A.11) thus implies that  $\chi$  is Lipschitz continuous with respect to the  $x$  variable and Hölder continuous with exponent  $\alpha$  with respect to the  $t$  variable. It is also Lipschitz continuous with respect to the  $z$  variable since it is in  $H^2(\mathbb{R}_z)$  and decays exponentially as well as its  $z$ -derivative. Hence  $n$  is in  $B^\alpha$  and

$$|n(t, x, z) - n(t', x', z')| \leq C|(t, x, z) - (t', x', z')|^\alpha$$

which leads, in view of (2.26), to

$$|n(t, x, z) - n(t', x', z')| \leq C|(t, x, z) - (t', x', z')|^{\lambda\alpha} (1 + |x| + |z|)^{-(1-\lambda)\gamma} \quad (2.27)$$

if  $|(x, z) - (x', z')| < 1/2$ . We are now able to apply Proposition 4.1 of Ref. [19] which insures that  $V \in B^{\lambda\alpha, 2+\lambda\alpha}(\mathbb{R}^3)$ . This implies the desired regularity for  $\epsilon$ , thanks to Lemma A.4.  $\square$

### 3 A priori estimates for the Vlasov-Schrödinger-Poisson system

#### 3.1 The energy estimate

In this section, we express the physical quantities which are conserved during the evolution of the system. For this purpose, we introduce different macroscopic quantities. The surface (*i.e.* integrated with respect to  $z$ ) charge density and surface current density (in direction  $x$ ) are respectively given by

$$\rho(t, x) = \int_{\mathbb{R}^2} f(t, x, v) dv \quad ; \quad j(t, x) = \int_{\mathbb{R}^2} v f(t, x, v) dv$$

and the charge density is

$$n(t, x, z) = \rho(t, x) |\chi(t, x, z)|^2.$$

The kinetic energy density is given by

$$\mathcal{E}_c(t, x, z) = |\chi(t, x, z)|^2 \int_{\mathbb{R}^2} \frac{v^2}{2} f(t, x, v) dv + \frac{1}{2} |\partial_z \chi(t, x, z)|^2 \rho(t, x)$$

and the potential energy density is written

$$\mathcal{E}_p(t, x, z) = \frac{1}{2} |\nabla_{x,z} V(t, x, z)|^2 + V_{ext}(z) n(t, x, z).$$

**Proposition 3.1** *Let  $(f, \chi, V)$  be a classical solution of (1.1)–(1.3), under Assumption 1.2. Then the total charge and the total energy of the system are conserved: for all  $t > 0$  we have*

$$\frac{d}{dt} \iint_{\mathbb{R}^3} n(t, x, z) dx dz = 0, \quad (3.1)$$

$$\frac{d}{dt} \iint_{\mathbb{R}^3} (\mathcal{E}_c(t, x, z) + \mathcal{E}_p(t, x, z)) dx dz = 0. \quad (3.2)$$

**Proof.** An integration of the Vlasov equation with respect to  $v$  gives the continuity equation

$$\partial_t \rho + \operatorname{div}_x j = 0,$$

which implies (3.1), since  $\iint n(t, x, z) dx dz = \int \rho(t, x) dx$ . Next, multiplying the Vlasov equation (1.1) by  $\frac{v^2}{2}$  and integrating over  $x$  and  $v$ , we obtain, after some integrations by parts,

$$\frac{d}{dt} \iint_{\mathbb{R}^4} \frac{v^2}{2} f dv dx - \int_{\mathbb{R}^2} \epsilon \operatorname{div}_x j dx = 0. \quad (3.3)$$

Thanks to the continuity equation, one can transform the second term as follows:

$$- \int_{\mathbb{R}^2} \epsilon \operatorname{div}_x j dx = \frac{d}{dt} \int_{\mathbb{R}^2} \rho \epsilon dx - \int_{\mathbb{R}^2} \rho \partial_t \epsilon dx.$$

Next we recall the two following identities:

$$\epsilon = \frac{1}{2} \langle |\partial_z \chi|^2 \rangle + \langle |\chi|^2 (V + V_{ext}) \rangle \quad ; \quad \partial_t \epsilon = \langle |\chi|^2 \partial_t (V + V_{ext}) \rangle.$$

Since  $n = \rho |\chi|^2$  and  $V_{ext}$  is independent of  $t$ , we deduce that

$$\begin{aligned} - \int_{\mathbb{R}^2} \epsilon \operatorname{div}_x j dx &= \frac{1}{2} \frac{d}{dt} \iint_{\mathbb{R}^3} \rho |\partial_z \chi|^2 dx dz + \frac{d}{dt} \iint_{\mathbb{R}^3} n V dx dz - \iint_{\mathbb{R}^3} n \partial_t V dx dz \\ &\quad + \frac{d}{dt} \iint_{\mathbb{R}^3} n V_{ext} dx dz. \end{aligned}$$

By the Poisson equation (1.3) we have

$$\iint_{\mathbb{R}^3} n V dx dz = \iint_{\mathbb{R}^3} |\nabla_{x,z} V|^2 dx dz \quad ; \quad \iint_{\mathbb{R}^3} n \partial_t V dx dz = \frac{1}{2} \frac{d}{dt} \iint_{\mathbb{R}^3} |\nabla_{x,z} V|^2 dx dz.$$

Inserting these equalities in (3.3) gives (3.2).  $\square$

### 3.2 $L^\infty$ estimate of the surface density

The construction of a classical solution for the coupled Vlasov-Schrödinger-Poisson system will rely on the following key estimate:

**Proposition 3.2** *Let  $(f, \chi, V)$  be a classical solution of (1.1)–(1.3), under Assumption (1.2). Then for all  $T > 0$  it satisfies the estimate:*

$$\sup_{t \in [0, T]} \|\rho(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq C_T,$$

where  $C_T$  is a constant depending only on  $T$  and on the data  $f_0$ .

**Proof.** Since  $\int |\chi|^2 dz = 1$ , the conservation of charge expressed by (3.1) and the fact that  $f$  is nonnegative imply

$$\|f(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^4)} = \|\rho(t, \cdot)\|_{L^1(\mathbb{R}^2)} = \|f_0\|_{L^1(\mathbb{R}^4)}.$$

Moreover, as a consequence of this energy estimate (3.2), the second order moment of the nonnegative  $f$  is bounded in  $L^1$ :

$$\iint_{\mathbb{R}^4} v^2 f(t, x, v) dv dx \leq C.$$

Together with the  $L^\infty$  estimate  $\|f(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^4)} = \|f_0\|_{L^\infty(\mathbb{R}^4)}$ , and a standard interpolation lemma in two dimensions, this leads to the following bound :

$$\|\rho(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq C. \tag{3.4}$$

Hence  $\rho$  is bounded in  $L^\infty((0, T), L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2))$  and by Lemma 2.4, we have for any  $t \in [0, T]$

$$\|\nabla_x \epsilon(t, \cdot)\|_\infty \leq C \|\rho(t, \cdot)\|_\infty^{1/2},$$

where  $C$  only depends on  $T$  and on the data of the problem. By applying Lemma B.2 given in the Appendix, thanks to Assumption 1.2, we get

$$\begin{aligned} \|\rho(t, \cdot)\|_\infty &\leq C \left( 1 + \int_0^t \|\nabla_x \epsilon(s, \cdot)\|_\infty^2 ds \right) \\ &\leq C \left( 1 + \int_0^t \|\rho(s, \cdot)\|_\infty ds \right). \end{aligned}$$

which yields the boundedness of  $\|\rho\|_{L_t^\infty L_x^\infty}$  after a Gronwall argument.  $\square$

## 4 Existence and uniqueness of the global classical solution

As mentioned in the Introduction, the construction of a solution is inspired by the work of Ukai and Okabe [19], who solved the Vlasov-Poisson system in dimension 2. The adaptation of this work relies on the fact that the Schrödinger-Poisson system studied in Section 2 shares similar regularization properties to the Poisson equation. The global existence of solutions of the two dimensional Vlasov-Poisson system relies on an  $L^\infty$  bound for the electric field which is obtained by a Gronwall argument. This argument fails in the Vlasov-Schrödinger-Poisson problem. The  $L^\infty$  bound holds however, but its proof relies on the use of the energy conservation satisfied by the solution (see Proposition 3.2). This fact induces some changes in the construction of the solution : namely, we are not able to construct the solution directly on arbitrarily large time intervals. At variance, the solution is constructed on small intervals, whose length only depend on some bounds on the initial data. The uniform bound (Proposition 3.2) on the solution allows to resume the construction on a new small interval, and so on. Since the length of the intervals stays away from zero, global existence is deduced.

In order to prove the local existence and uniqueness, we introduce the following Vlasov-Poisson like system (in dimension 2 here, but this can easily be generalized):

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{F}[\rho] \cdot \nabla_v f = 0, \\ \rho(t, x) = \int_{\mathbb{R}^2} f(t, x, v) dv \\ f(0, x, v) = f_0(x, v), \end{cases} \quad (4.1)$$

where the functional

$$\rho(t, \cdot) \mapsto \mathcal{F}[\rho(t, \cdot)] \in L^\infty(\mathbb{R}^2)^2,$$

local in time, is defined on  $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$  and satisfies the four following properties:

**(P1)** for any  $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$  we have

$$\|\mathcal{F}[\rho]\|_\infty \leq C(\|\rho\|_1) \|\rho\|_\infty^\beta,$$

where  $\beta$  is a positive real number and  $C(\|\rho\|_1)$  is a constant which only depends on  $\|\rho\|_1$ ;

**(P2)** if  $\rho$  and  $\tilde{\rho}$  belong to  $L^1 \cap L^\infty$ , then

$$\|\mathcal{F}[\rho] - \mathcal{F}[\tilde{\rho}]\|_\infty \leq C_{\rho, \tilde{\rho}}(\|\rho - \tilde{\rho}\|_1 + \|\rho - \tilde{\rho}\|_\infty),$$

where  $C_{\rho, \tilde{\rho}}$  only depends on the  $L^1$  and  $L^\infty$  norms of  $\rho$  and  $\tilde{\rho}$ ;

**(P3)** if  $\rho = \rho(t, x)$  belongs to  $B^\alpha([0, T] \times \mathbb{R}^2)$ , where  $0 < \alpha < 1$ , and satisfies (2.25) then there exists  $\alpha_1 > 0$  and  $\alpha_2 > 0$  such that  $\mathcal{F}[\rho]$  belongs to  $B^{\alpha_1, 1+\alpha_2}(\mathbb{R}^2)$ ;

**(P4)** for any  $x, x' \in \mathbb{R}^d$ , we have

$$|\mathcal{F}[\rho](x) - \mathcal{F}[\rho](x')| \leq C_\rho \zeta(|x - x'|),$$

where  $C_\rho$  only depends on  $\|\rho\|_1 + \|\rho\|_\infty$  and the function  $\zeta$  was defined in Theorem 2.1.

Under these assumptions, we have the following local existence result:

**Proposition 4.1** *Let  $\mathcal{F}$  satisfy Properties **(P1)**, **(P2)**, **(P3)**, **(P4)** and  $f_0$  satisfy Assumption 1.2. Then (4.1) admits a unique classical solution on a maximal time interval  $[0, T_0)$  such that  $T_0 \in (0, +\infty)$ . Moreover if  $T_0 < +\infty$  then we have*

$$\sup_{t \in [0, T_0)} \|\rho(t, \cdot)\|_\infty = +\infty.$$

**Proof.** We shall first prove the existence of a local in time solution, then show the uniqueness of this solution.

**Existence.** The construction of a local in time solution is done by adapting the work of Ukai and Okabe [19] for the two dimensional Vlasov-Poisson system. We first define the subset  $S$  of  $B^0([0, T] \times \mathbb{R}^4)$ , consisting of all the functions  $g$  which satisfy the following conditions:

- (i)  $g \in B^{\alpha_0}([0, T] \times \mathbb{R}^4)$ ,
- (ii)  $\|g\|_{B^{\alpha_0}} \leq M_0$ ,
- (iii)  $|g(t, x, v)| \leq M_0(1 + |x|)^{-\gamma} (1 + |v|)^{-\gamma}$ ,  $(t, x, v) \in [0, T] \times \mathbb{R}^4$ ,
- (iv)  $\iint_{\mathbb{R}^4} |g(t, x, v)| dx dv \leq \|f_0\|_1$ ,  $t \in [0, T]$ ,
- (v)  $\int_{\mathbb{R}^2} |g(t, x, v)| dv \leq M_1(t)$ ,  $(t, x) \in [0, T] \times \mathbb{R}^2$ ,

where the positive constants  $T$ ,  $M_0$  and  $\alpha_0$ , as well as the positive nondecreasing function  $M_1(t)$ , will be precised further. It is readily seen that  $S$  is a compact convex subset of  $B^0([0, T] \times \mathbb{R}^4)$ . For any element  $g \in S$ , we define  $f := \Gamma(g)$  as the solution of

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{F}[\rho_g] \cdot \nabla_v f = 0, \\ \rho_g(t, x) = \int_{\mathbb{R}^d} g(t, x, v) dv \\ f(0, x, v) = f_0(x, v). \end{cases} \quad (4.2)$$

Thanks to the properties **(P1)** to **(P4)** of  $\mathcal{F}$ , one can apply Lemma B.1 of the Appendix. Remark that if  $g$  satisfies (i) and (iii) with  $\gamma > 2$ , then  $\rho_g \in B^\alpha$  for some  $0 < \alpha < \alpha_0$ , thus by **(P3)**, the force field  $\mathcal{F}[\rho_g]$  belongs to  $C^0([0, T], C^1 \cap L^\infty(\mathbb{R}^2))$ .

For any  $g \in S$  the function  $\Gamma(g)$  is uniquely defined, continuously differentiable, and satisfies the estimates (B.2), (B.3) and (B.4). In addition, **(P1)**, **(P4)** and (iv)

imply that the constants  $\alpha(F)$  and  $\mathcal{M}(F)$  appearing in these estimates depend only on  $\|\rho\|_\infty$  : we write these constants  $\alpha(\|\rho\|_\infty)$  and  $\mathcal{M}(\|\rho\|_\infty)$  (the functions  $\alpha(\cdot)$  and  $\mathcal{M}(\cdot)$  are respectively nonincreasing and nondecreasing).

Consequently, the crucial step is to get an estimate for  $\|\rho\|_\infty$ . This can be done by using Lemma B.2 and **(P1)**. Let  $\rho_f = \int f dv$  and  $\rho_g = \int g dv$ , where  $f$  solves (4.2). We have

$$\|\rho_f(t, \cdot)\|_\infty \leq C_0 \left( 1 + \int_0^t \|\rho_g(s, \cdot)\|^{2\beta} ds \right). \quad (4.3)$$

Let  $\rho_0$  be the positive nondecreasing solution of the integral equation

$$\rho_0(t) = C_0 \left( 1 + \int_0^t \rho_0(s)^{2\beta} ds \right).$$

This solution is defined on a maximal interval  $[0, T_0)$ . Let  $0 < T < T_0$ . Now we fix the constants in (i)–(v) as follows:

$$\alpha_0 = \alpha(\rho_0(T)) \quad ; \quad M_0 = \mathcal{M}(\rho_0(T)) \quad ; \quad M_1(t) = \rho_0(t).$$

If  $g$  belongs to  $S$  then we have  $|\rho_g(t, x)| \leq \rho_0(T)$ , thus  $\alpha(\|\rho\|_\infty) \geq \alpha_0$  and  $\mathcal{M}(\|\rho\|_\infty) \leq M_0$ : it is clear by (B.2)–(B.4) that  $f = \Gamma(g)$  also satisfies (i), (ii), (iii) and (iv). Next, since  $g$  satisfies (v) and by (4.3), we get

$$\|\rho_f(t, \cdot)\|_\infty \leq C_0 \left( 1 + \int_0^t \rho_0(s)^{2\beta} ds \right) = \rho_0(t) = M_1(t);$$

this shows that  $f$  also satisfies (v). The set  $S$  is thus stable by  $\Gamma$ .

Then, with property **(P2)**, one can prove that  $\Gamma$  is continuous in the  $B^0$  topology (see [19]). This is enough to conclude that it admits a fixed point, which is solution of (4.1) on a time interval  $[0, T_0)$ . If  $\sup_{t \in [0, T_0)} \|\rho(t, \cdot)\|_\infty < +\infty$  then, by (B.2), (B.3) and **(P3)**, the solution is in fact defined on the closed interval  $[0, T_0]$ , and one can extend this solution after  $T_0$ . This proves the second part of the Proposition.

**Uniqueness.** It remains to prove the uniqueness of the solution of (4.1). Let  $f^1$  and  $f^2$  be two classical solutions of (4.1) and let  $\rho^1$  and  $\rho^2$  be the densities corresponding to  $f^1$  and  $f^2$ . Let us also use the notation

$$\|f\| = \|f\|_{L^1_{x,v}} + \|f\|_{L^\infty_x L^1_v}.$$

By differentiating (4.1) with respect to  $x$  or with respect to  $v$ , one gets

$$\|\nabla_x f^1(t, \cdot, \cdot)\|_1 \leq \|\nabla_x \mathcal{F}[\rho^1]\|_\infty \int_0^t \|\nabla_v f^1(s, \cdot, \cdot)\|_1 ds,$$

$$\|\nabla_v f^1(t, \cdot, \cdot)\|_1 \leq \int_0^t \|\nabla_x f^1(s, \cdot, \cdot)\|_1 ds,$$

thus **(P3)** and a Gronwall lemma imply that  $\nabla_x f^1$  and  $\nabla_v f^1$  are bounded in  $L^\infty((0, T), L^1 \cap L^\infty(\mathbb{R}^4))$ , as well as  $\nabla_x f^2$  and  $\nabla_v f^2$ .

Additionally, the difference  $f = f^1 - f^2$  satisfies the equation

$$\partial_t f + v \cdot \nabla_x f + \mathcal{F}[\rho^1] \cdot \nabla_v f = (\mathcal{F}[\rho^2] - \mathcal{F}[\rho^1]) \cdot \nabla_v f^2. \quad (4.4)$$

Therefore one deduces from Liouville's theorem that

$$\begin{aligned} \iint_{\mathbb{R}^4} |f(t, x, v)| dx dv &\leq \int_0^t \iint_{\mathbb{R}^4} |\mathcal{F}[\rho^2] - \mathcal{F}[\rho^1]|(s, x) |\nabla_v f^2(s, x, v)| dx dv ds \\ &\leq \|\nabla_v f^2\|_{L_t^\infty L_{x,v}^1} \int_0^t \|\mathcal{F}[\rho^2] - \mathcal{F}[\rho^1]\|_\infty(s) ds. \end{aligned}$$

Thanks to Property **(P2)**, we get

$$\|f^2(t) - f^1(t)\|_1 \leq C \|\nabla_v f^2\|_{L_t^\infty L_{x,v}^1} \int_0^t (\|\rho^2(s) - \rho^1(s)\|_1 + \|\rho^2(s) - \rho^1(s)\|_\infty) ds,$$

which implies

$$\|f^2(t) - f^1(t)\|_1 \leq C \int_0^t \|f^2(s) - f^1(s)\| ds.$$

Besides, the solution  $f$  of (4.4) reads

$$f(t, x, v) = \int_0^t \{(\mathcal{F}[\rho^2] - \mathcal{F}[\rho^1]) \cdot \nabla_v f^2\}(s, \mathcal{X}^1(s; t, x, v), \mathcal{V}^1(s; t, x, v)) ds, \quad (4.5)$$

where the characteristics are given by ( $i = 1, 2$ ):

$$\frac{d\mathcal{X}^i}{ds} = \mathcal{V}^i, \quad \frac{d\mathcal{V}^i}{ds} = \mathcal{F}[\rho^i](s, \mathcal{X}^i), \quad \mathcal{X}^i(t; t, x, v) = x, \quad \mathcal{V}^i(t; t, x, v) = v; \quad i = 1, 2.$$

Thanks to the boundedness of  $\mathcal{F}[\rho^1]$  in some  $B^{\alpha_1, 1+\alpha_2}$ , uniform bounds on  $[0, T]$  for  $\mathcal{X}^1$ ,  $\nabla_x \mathcal{X}^1$  and  $\nabla_v \mathcal{X}^1$  are easily obtained. Hence, from (1.6) satisfied by the initial data, one can deduce (see e.g. [19]) that

$$|\nabla_v f^2(s, \mathcal{X}^1(s; t, x, v), \mathcal{V}^1(s; t, x, v))| \leq C(1 + |v|)^{-\gamma}.$$

Integrating (4.5) with respect to  $v$ , we obtain

$$\|f^2(t) - f^1(t)\|_{L_x^\infty L_v^1} \leq C \int_0^t \|\mathcal{F}[\rho^2] - \mathcal{F}[\rho^1]\|_\infty(s) ds \leq C \int_0^t \|f^2(s) - f^1(s)\| ds.$$

By summing the above inequalities, we obtain

$$\|f^2(t) - f^1(t)\| \leq C \int_0^t \|f^2(s) - f^1(s)\| ds$$

which leads to  $f_2 \equiv f_1$ . □

**Application: proof of Theorem 1.3.** To prove Theorem 1.3, we only need to collect the results obtained in the previous sections. In Section 2, we studied the mapping  $\rho \mapsto \mathcal{F}[\rho] := -\nabla_x \epsilon$ , where  $\epsilon$  solves the quasistatic Schrödinger-Poisson system (2.1), (2.2). Properties **(P1)**, **(P2)**, **(P3)** and **(P4)** come respectively from (2.4), (2.6), Lemma 2.5 and (2.5) (note that we have  $\beta = 5/6$  in **(P1)**).

Then the first part of Proposition 4.1 gives the existence of a unique classical solution on a maximal time interval  $[0, T_0)$ . Finally, the global estimate given in Proposition 3.2 and the second part of Proposition 4.1 show that  $T_0 = +\infty$ .  $\square$

## Appendix

### A Properties of the Schrödinger eigenvalue problem

In this part, we recall some spectral properties of the operator  $H[V] = -\frac{1}{2}\frac{d^2}{dz^2} + (V + V_{ext})$  on  $L^2(\mathbb{R})$  with the domain (independent of  $V$ )

$$D(H) = \{ \psi \in H^2(\mathbb{R}) : V_{ext} \psi(z) \in L^2(\mathbb{R}) \}.$$

We assume here that  $V_{ext}$  satisfies Assumption 1.1 and that  $V$  belongs to  $L^\infty(\mathbb{R})$ . It is well-known (see for instance [16]) that this operator is self-adjoint, bounded from below and has a compact resolvent. Its eigenfunctions  $(\chi_p)_{p \in \mathbb{N}^*}$ , chosen real-valued, form an orthonormal basis of  $L^2(\mathbb{R})$  and its eigenvalues  $(\epsilon_p)_{p \in \mathbb{N}^*}$  form a strictly increasing sequence of real numbers tending to infinity. Moreover we have

$$\forall a > 0 \quad \forall p \in \mathbb{N}^* \quad \exists C_{a,p,V} > 0 \quad \text{such that,} \quad \forall z \in \mathbb{R}, \quad |\chi_p(z)| \leq C_{a,p,V} e^{-a|z|}, \quad (\text{A.1})$$

where  $C_{a,p,V}$  only depends on  $a$ ,  $p$ ,  $V_{ext}$  and  $\|V\|_\infty$ . The  $p$ -th eigenvalue is also given by the max-min formula

$$\epsilon_p[U] = \max_{\dim E_p = p-1} \min_{\substack{\phi \in E_p^\perp \cap D(H) \\ \|\phi\|_{L^2(\mathbb{R})} = 1}} \left( \int_{\mathbb{R}} \frac{1}{2} \left| \frac{d\phi}{dz}(z) \right|^2 dz + \int_{\mathbb{R}} (U + V_{ext})(z) |\phi(z)|^2 dz \right). \quad (\text{A.2})$$

Here are two immediate consequences of this formula, where  $U$  and  $V$  are two functions in  $L^\infty(\mathbb{R})$ :

$$\text{if } U \geq V \text{ a.e. on } \mathbb{R} \quad \text{then} \quad \forall p \in \mathbb{N}^* \quad \epsilon_p(U) \geq \epsilon_p(V). \quad (\text{A.3})$$

$$|\epsilon_p[U] - \epsilon_p[V]| \leq \|U - V\|_\infty. \quad (\text{A.4})$$

Next, from

$$\epsilon_p[V] = \int_{\mathbb{R}} \frac{1}{2} \left| \frac{d\chi_p}{dz}(z) \right|^2 dz + \int_{\mathbb{R}} (V + V_{ext})(z) |\chi_p(z)|^2 dz \quad (\text{A.5})$$

and (A.4) with  $U = 0$ , we obtain

$$\|\chi_p[V]\|_{H^1(\mathbb{R})}^2 \leq C_p + \|V\|_{L^\infty(\mathbb{R})}.$$

Therefore, by a Gagliardo-Nirenberg inequality, we deduce that

$$\|\chi_p[V]\|_{L^\infty(\mathbb{R})} \leq C_p (1 + \|V\|_{L^\infty(\mathbb{R})}^{1/4}). \quad (\text{A.6})$$

The following differentiability result is standard:

**Lemma A.1** *For any  $p \in \mathbb{N}^*$ , the mapping  $V \mapsto \epsilon_p[V]$  is differentiable on  $L^\infty(\mathbb{R})$  and we have*

$$\forall W \in L^\infty(\mathbb{R}) \quad d_V \epsilon_p(W) = \langle W | \chi_p|^2 \rangle.$$

The following lemma, also given without proof, contains additional information on the differential of the eigenfunctions and eigenvalues when the potential depends on a parameter:

**Lemma A.2** *Let  $V = V(x, z) \in L^\infty(\mathbb{R}^3)$ . Let us denote  $\epsilon_p(x)$  instead of  $\epsilon_p[V(x, \cdot)]$  and analogously for  $\chi_p(x, \cdot)$ . Assume that  $\nabla_x V \in L_{loc}^1(\mathbb{R}^2, L^\infty(\mathbb{R}))$ .*

(i) *Then  $\nabla_x \epsilon_p \in L_{loc}^1(\mathbb{R}^2)$  and we have*

$$\nabla_x \epsilon_p = \langle |\chi_p|^2 \nabla_x V \rangle \quad (\text{A.7})$$

$$|\nabla_x \epsilon_p(x)| \leq C \|\nabla_x V(x, \cdot)\|_{L^\infty(\mathbb{R})}, \quad (\text{A.8})$$

where  $C$  is independent of  $V$ .

(ii) *Furthermore, there exists a parametrization  $\chi_p(x, \cdot)$  such that  $\nabla_x \chi_p$  belongs to  $L_{loc}^1(\mathbb{R}^2, L^\infty(\mathbb{R}))$  and we have*

$$\nabla_x \chi_p = \sum_{q \neq p} \frac{\langle \chi_p \chi_q \nabla_x V \rangle}{\epsilon_p - \epsilon_q} \chi_q, \quad (\text{A.9})$$

$$\|\nabla_x \chi_p(x, \cdot)\|_{L^\infty(\mathbb{R})} \leq C_V \|\nabla_x V(x, \cdot)\|_{L^\infty(\mathbb{R})}, \quad (\text{A.10})$$

where the constant  $C_V$  only depends on  $\|V\|_{L^\infty(\mathbb{R}^3)}$  and not on the index  $p$ .

From this lemma one can deduce the following

**Corollary A.3** (i) *Let  $V$  and  $\tilde{V}$  be in  $L^\infty(\mathbb{R})$ . Then for any  $p \in \mathbb{N}^*$ , the corresponding  $p$ -th eigenfunctions satisfy*

$$\|\chi_p[V] - \chi_p[\tilde{V}]\|_{L^\infty(\mathbb{R})} \leq C_{V, \tilde{V}} \|V - \tilde{V}\|_{L^\infty(\mathbb{R})}. \quad (\text{A.11})$$

(ii) *Let  $V$  and  $\tilde{V}$  be in  $W^{1, \infty}(\mathbb{R}^2, L^\infty(\mathbb{R}))$ . Then the corresponding eigenvalues satisfy*

$$\|\nabla_x \epsilon_p[V] - \nabla_x \epsilon_p[\tilde{V}]\|_{L^\infty(\mathbb{R}^3)} \leq C_{V, \tilde{V}} \|\nabla_x V - \nabla_x \tilde{V}\|_{L^\infty(\mathbb{R}^3)}. \quad (\text{A.12})$$

In this lemma, the constant  $C_{V, \tilde{V}}$  only depends on  $p$ ,  $\|V\|_\infty$  and  $\|\tilde{V}\|_\infty$ .

We end this section by the following Lemma –also standard and stated without proof– which says that  $\epsilon$  has the same regularity as  $V$  with respect to  $x$ :

**Lemma A.4** *If  $V \in B^{k+\alpha}(\mathbb{R}^2, L^\infty(\mathbb{R}))$ , for  $k \in \mathbb{N}$  and  $0 \leq \alpha < 1$ , then the corresponding eigenvalues  $\epsilon_p[V]$  belong to  $B^{k+\alpha}(\mathbb{R}^2)$  and the map  $V \mapsto \epsilon_p[V]$  is locally Lipschitz continuous in these spaces.*

## B Results concerning the Vlasov and the Poisson equations

Consider the Vlasov equation in dimension  $d$

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \\ f(0, x, v) = f_0(x, v), \end{cases} \quad (\text{B.1})$$

where  $F(t, x)$  denotes a generic force field defined on  $[0, T] \times \mathbb{R}^d$  and belonging to  $C^0([0, T], C^1(\mathbb{R}^d))$ . Then, if  $f_0$  is continuously differentiable, (B.1) admits a unique classical solution  $f \in C^1([0, T] \times \mathbb{R}^{2d})$ . This solution can be written simply thanks to the characteristic equations, which are defined by  $\mathcal{X}(s; t, x, v)$ ,  $\mathcal{V}(s; t, x, v)$  solving

$$\frac{d\mathcal{X}}{ds} = \mathcal{V}, \quad \frac{d\mathcal{V}}{ds} = F(s, \mathcal{X}), \quad \mathcal{X}(t; t, x, v) = x, \quad \mathcal{V}(t; t, x, v) = v.$$

Indeed, we have

$$f(t, x, v) = f_0(\mathcal{X}(0; t, x, v), \mathcal{V}(0; t, x, v)).$$

The following lemma is taken from [19, Proposition 6.1–Lemma 6.1]:

**Lemma B.1** *Let  $F(t, x)$  be in  $C^0([0, T], C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$  and let*

$$\mu = \sup_{(t, x, x') \in [0, T] \times \mathbb{R}^{2d}} \frac{|F(t, x) - F(t, x')|}{\zeta(|x - x'|)},$$

where the function  $\zeta$  was defined in Theorem 2.1. Assume that  $f_0$  satisfies Assumption 1.2. Then there exists a nonincreasing function  $\alpha(\cdot)$  and a nondecreasing function  $\mathcal{M}(\cdot)$  such that the solution of (B.1) satisfies the following estimates:

$$\|f\|_{B^{\alpha(F)}} \leq \mathcal{M}(F), \quad (\text{B.2})$$

$$|f(t, x, v)| \leq \mathcal{M}(F) (1 + |x|)^{-\gamma} (1 + |v|)^{-\gamma}, \quad (\text{B.3})$$

$$\iint_{\mathbb{R}^4} |f(t, x, v)| dx dv = \|f_0\|_1. \quad (\text{B.4})$$

where we have denoted for simplicity  $\alpha(F) = \alpha(\mu + \|F\|_\infty)$  and  $\mathcal{M}(F) = \mathcal{M}(\mu + \|F\|_\infty)$ .

Another classical result is that the  $L^\infty$  norm of the density is controlled by the  $L^\infty$  norm of the field. Denoting  $\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv$ , we have the

**Lemma B.2** *Let  $f_0$  satisfy*

$$|f_0(x, v)| \leq C (1 + |x|)^{-\gamma} (1 + |v|)^{-\gamma},$$

where  $\gamma > d$ . Then the solution of (B.1) satisfies

$$|\rho(t, x)| \leq C \left( 1 + \int_0^t \|F(s, \cdot)\|_\infty^d ds \right).$$

**Proof.** The proof is standard and can be found for example in [19].  $\square$

The following lemma gives some estimates on the solution of the Poisson equation in dimension 3 (see for instance [1]):

**Lemma B.3** *Let  $f \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ , with  $3/2 < p \leq \infty$ . Then we have*

$$\left\| \frac{1}{r} * f \right\|_{\infty} \leq C \|f\|_p^{\theta} \|f\|_1^{1-\theta}, \quad (\text{B.5})$$

with  $\theta = \frac{p}{3p-3}$ . Moreover, if  $p = \infty$  we have

$$\left\| \nabla_{x,z} \left( \frac{1}{r} * f \right) \right\|_{\infty} \leq C \|f\|_{\infty}^{2/3} \|f\|_1^{1/3}. \quad (\text{B.6})$$

We conclude this appendix by the following Lemma which was proven in [4]. It deals with the convolution in dimension 3 of the Poisson kernel  $\frac{1}{4\pi r}$  with  $L_x^p L_z^1$  densities:

**Lemma B.4** (i) *Let  $f \in L_x^p L_z^1$  with  $1 < p < 2$ . Then we have*

$$\left\| \frac{1}{r} * f \right\|_{p^{\#}, \infty} + \left\| \nabla_{x,z} \left( \frac{1}{r} * f \right) \right\|_{p^{\#}, 1} \leq C_p \|f\|_{p,1}, \quad (\text{B.7})$$

where  $p^{\#} = \frac{2p}{2-p}$ .

(ii) *Let  $f \in L_x^p L_z^1 \cap L^1(\mathbb{R}^3)$  with  $2 < p \leq \infty$ . Then we have*

$$\left\| \frac{1}{r} * f \right\|_{\infty} + \left\| \nabla_{x,z} \left( \frac{1}{r} * f \right) \right\|_{\infty, 1} \leq C_p \|f\|_{p,1}^{\theta} \|f\|_1^{1-\theta}, \quad (\text{B.8})$$

where  $\theta = \frac{p}{2p-2}$ .

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