

Structure of the linearized gravitational Vlasov-Poisson system close to a polytropic ground state

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Abstract

We deal in this paper with a generalized gravitational Vlasov-Poisson system that covers the three and four dimensional cases as well as the three dimensional ultra-relativistic case. This system admits polytropic stationary solutions which are orbitally stable, see [14], [22], [37]. We study in this paper the linear system obtained after a linearization close to these ground states and prove that the linearized flow displays at most algebraic instabilities. The heart of the proof is the derivation of a positivity property for the linearized Hamiltonian that implies a “quantitative” proof of the orbital stability statement. Our strategy follows the analysis by M. Weinstein [43], who obtained similar results for the nonlinear Schrödinger equation that turned out to be fundamental preliminary properties for the further description of the fine qualitative properties of the Hamiltonian system.

1 Introduction

1.1 The gravitational Vlasov-Poisson system

We consider in this paper the following generalized gravitational Vlasov-Poisson system:

$$(VP) \quad \begin{cases} \partial_t f + |v|^{\alpha-2} v \cdot \nabla_x f - E_f \cdot \nabla_v f = 0, & (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N \\ f(t=0, x, v) = f_0(x, v) \geq 0, \end{cases} \quad (1.1)$$

in the range of parameters

$$(N, \alpha) \in \{(3, 1), (3, 2), (4, 2)\} \quad (1.2)$$

and where we denoted for a given distribution $f \geq 0$:

$$E_f(x) = \nabla_x \phi_f, \quad \phi(x) = -\frac{1}{N(N-2)\omega_N} \frac{1}{|x|^{N-2}} \star \rho_f, \quad \rho_f(x) = \int_{\mathbb{R}^N} f(x, v) dv, \quad (1.3)$$

ω_N being the volume of the unit ball in \mathbb{R}^N ($\omega_3 = \frac{4\pi}{3}$ and $\omega_4 = \frac{\pi^2}{2}$). Our range of parameters (1.2) covers the three following situations:

- The three dimensional gravitational Vlasov-Poisson system $(N, \alpha) = (3, 2)$ which describes the mechanical state of a stellar system subject to its own gravity (see for instance [4, 9]) and whose classical solutions are global in time, see Lions and Perthame [28], Pfaffelmoser [33], Schaeffer [38].
- Classical calculations show that this model should be correct only for low velocities and if high velocities occur, special relativistic corrections should be introduced, see Van Kampen and Felderhof [41], Glassey and Schaeffer [11, 12]. A more accurate model is then provided by the relativistic three dimensional Vlasov-Poisson system:

$$(RVP) \quad \begin{cases} \partial_t f + \frac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f - E_f \cdot \nabla_v f = 0, & (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \\ f(t=0, x, v) = f_0(x, v) \geq 0. \end{cases} \quad (1.4)$$

A major difference with the three dimensional (VP) is that this system may develop finite time blow up singularities, see [10, 11], and a preliminary model problem is given by the three dimensional ultrarelativistic (VP) system which is (1.1) with $(N, \alpha) = (3, 1)$.

- The four dimensional Vlasov-Poisson system $(N, \alpha) = (4, 2)$ is a fundamental mathematical model for the study of the singularity formation, see [22, 23], which shares a similar critical structure like (1.4) but admits extra fundamental invariances, and in particular an explicit pseudo-conformal symmetry.

A natural space to study the (VP) system is the energy space

$$\mathcal{E} = \{f \geq 0 \text{ with } |f|_{\mathcal{E}} = |f|_{L^1} + |f|_{L^p} + ||v|^\alpha f|_{L^1} < +\infty\}$$

for $p_{crit} < p < +\infty$ where:

$$p_{crit} = \frac{N\alpha + (\alpha + 2)N - N^2}{2\alpha + (\alpha + 2)N - N^2} = \begin{cases} 9/7 & \text{for } (N, \alpha) = (3, 2), \\ 2 & \text{for } (N, \alpha) = (4, 2), \\ 3/2 & \text{for } (N, \alpha) = (3, 1). \end{cases} \quad (1.5)$$

Recall that (1.1) satisfies *formally* some conservation laws: for all $q \in [1, p]$ the L^q norm of a solution f is independent of time, as well as the Hamiltonian defined by

$$\mathcal{H}(f) = \frac{1}{\alpha} \int_{\mathbb{R}^{2N}} |v|^\alpha f - \frac{1}{2} \int_{\mathbb{R}^N} |E_f|^2. \quad (1.6)$$

Moreover, a large group of symmetries leaves (1.1) invariant: if $f(t, x, v)$ solves (1.1), then $\forall (t_0, x_0, \lambda_0, \mu_0) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}_+^* \times \mathbb{R}_+^*$, so does:

$$\frac{\mu_0^{N-\alpha}}{\lambda_0^2} f \left(\frac{t+t_0}{\lambda_0 \mu_0^{\alpha-1}}, \frac{x+x_0}{\lambda_0}, \mu_0 v \right). \quad (1.7)$$

The case $\alpha = 2$ also enjoys the Galilean invariance: if $f(t, x, v)$ solves (1.1), then $\forall v_0 \in \mathbb{R}^N$, so does $f(t, x + v_0 t, v + v_0)$.

In the classical case corresponding to $\alpha = 2$, the existence of weak solutions for (1.1) in the energy space \mathcal{E} is due to Horst and Hunze [19] and Diperna and Lions [7, 8]. These solutions verify an upper bound on the Hamiltonian

$$\mathcal{H}(f(t)) \leq \mathcal{H}(f_0) \quad (1.8)$$

and the exact conservation of the L^q norm

$$\forall 1 \leq q \leq p, \quad |f(t)|_{L^q} = |f_0|_{L^q}. \quad (1.9)$$

In the ultrarelativistic case $\alpha = 1$, we are not aware of any result concerning the Cauchy theory for (1.1).

In the energy space, we have the interpolation estimate:

$$\forall f \in \mathcal{E} \quad |E_f|_{L^2}^2 \leq C_p \|v\|^\alpha |f|_{L^1}^{\theta_1} |f|_{L^p}^{\theta_2} |f|_{L^1}^{\theta_3} \quad (1.10)$$

with

$$\theta_1 = \frac{N-2}{\alpha}, \quad \theta_2 = \frac{(N-2)p}{N(p-1)}, \quad \theta_3 = 2 - \theta_1 - \theta_2. \quad (1.11)$$

Note that we have $0 < \theta_i < 2$, for $i = 1, 2, 3$ in the range of parameters (1.2) and $p_{crit} < p < +\infty$. In particular, for $(N, \alpha) = (3, 2)$, $\theta_1 = \frac{1}{2}$ and thus the bound on the Hamiltonian (1.8) and the conservation of the L^1, L^p norms imply a uniform bound on the kinetic energy, hence the existence of a global weak solution to (1.1); on the contrary for $(N, \alpha) \in \{(3, 1), (4, 2)\}$, $\theta_1 = 1$ and blow up can indeed occur from a classical virial identity, see [11]. The blow up problem in this case is of critical nature in the sense that the strength of the kinetic and the potential energy in the Hamiltonian is the same, see [22] for a further discussion on this problem.

1.2 Linear and nonlinear stability

Our aim in this paper is to study the properties of the linear flow close to a specific class of stationary solutions, the so-called polytropic ground states. This is a classical problematic related to the question of the linear and nonlinear stability of the stationary solutions which has been addressed in a number of works for the case of the three dimensional gravitational Vlasov Poisson system (1.1) for $(N, \alpha) = (3, 2)$.

A large class of stationary solutions to (1.1) for $(N, \alpha) = (3, 2)$ of the form $f(t, x, v) = F(e)$ where $e = \frac{|v|^2}{2} + \phi(x)$, has been constructed in [2] by solving the associated non linear radial ODE. Two classical strategies then emerge to prove the nonlinear stability of such solutions: variational techniques for those stationary solutions than can be obtained as minimizers of a well chosen functional; direct linearization techniques using the conservation of the Hamiltonian and coercivity properties of the linearized energy.

The first approach has been used in particular by Guo and Rein, [13, 14, 15, 35, 16] where part of these steady states including the polytropes have been obtained as minimizers of appropriately chosen energy-Casimir functionals under a constraint of prescribed mass. As observed in [22], see also Sanchez and Soler [37], a direct application of the original concentration technique introduced by P.-L. Lions in [26], [27] allows one to recover the orbital stability of a two parameters family of ground states –while the Energy-Casimir technique only covers one parameter families– in the energy space by proving the strong relative compactness up to space translation of the minimizing sequences of the problem:

$$\min_{f \geq 0, |f|_{L^1} = M_1, |j(f)|_{L^1} = M_j} \mathcal{H}(f) \quad (1.12)$$

for a large class of convex functions j . Note that the two parameters family is in correspondence with the two parameters scaling invariance of the Vlasov-Poisson system. Here a difficulty arises however which is that uniqueness for (1.12) is known only in two special cases: when $j(f) = f^p$ which is the case of polytropes where uniqueness follows directly from the scaling invariance of the polytropic equation (1.17); when the minimizer of (1.12) is also a minimizer of the following one constraint minimization problem:

$$\min_{f \geq 0, \|f\|_{L^1} + \|j(f)\|_{L^1} = M} \mathcal{H}(f) \quad (1.13)$$

– which can easily be proved to hold for a large subclass of solutions to (1.12) – and then one may use Schaeffer’s uniqueness result [39]. Using extra scaling invariances in the case of the polytrope, we have the following result which was proved for $\alpha = 2$ and $N = 3, 4$, in [22] and easily adapts to $(N, \alpha) = (3, 1)$:

Proposition 1.1 (Variational characterization of the ground state, [22]) *Let (N, α) satisfy (1.2), $p \in (p_{crit}, +\infty)$ and $(\theta_i)_{1 \leq i \leq 3}$ given by (1.11). The minimization problem*

$$\inf_{f \in \mathcal{E}, f \neq 0} \frac{\|v\|^\alpha \|f\|_{L^1}^{\theta_1} \|f\|_{L^p}^{\theta_2} \|f\|_{L^1}^{\theta_3}}{\|E_f\|_{L^2}^2} \quad (1.14)$$

is attained on the four parameters family

$$\gamma Q \left(\frac{x - x_0}{\lambda}, \mu v \right), \quad (\gamma, \lambda, \mu, x_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}^N. \quad (1.15)$$

Here Q is the polytropic ground state:

$$Q_{\alpha, p, N}(x, v) = \begin{cases} \left(-1 - \frac{|v|^\alpha}{\alpha} - \phi_Q(x) \right)^{\frac{1}{p-1}} & \text{for } \frac{|v|^\alpha}{\alpha} + \phi_Q(x) < -1, \\ 0 & \text{for } \frac{|v|^\alpha}{\alpha} + \phi_Q(x) > -1. \end{cases} \quad (1.16)$$

where ϕ_Q is the unique non trivial radial solution to:

$$-\frac{1}{r^{N-1}} \frac{d}{dr} (r^{N-1} \phi_Q') + \gamma_{\alpha, p, N} (-1 - \phi_Q)_+^{\frac{1}{p-1} + \frac{N}{\alpha}} = 0, \quad \phi(r) \rightarrow 0 \quad \text{as } r \rightarrow +\infty, \quad (1.17)$$

$\gamma(\alpha, p, N)$ given by

$$\gamma_{\alpha, p, N} = \sigma_N \int_0^1 (\alpha t)^{\frac{N-\alpha}{\alpha}} (1-t)^{\frac{1}{p-1}} dt. \quad (1.18)$$

For $(N, \alpha) = (3, 2)$, Q is moreover orbitally stable in the energy space by the flow of (1.1) and orbital stability up to an additional scaling invariance holds as well for $(N, \alpha) = (4, 2)$, see [22], Sanchez and Soler [37], see also Hadzic [18].

Theorem 1.2 (Orbital stability of the ground state for $(N, \alpha) = (3, 2)$) *Let $(N, \alpha) = (3, 2)$ and $p_{crit} < p < +\infty$. Then for all $\eta > 0$, there exists $\delta(\eta) > 0$ such that the following holds true. Let $f_0 \in \mathcal{E}$ with*

$$\mathcal{H}(f_0) - \mathcal{H}(Q) \leq \delta(\eta), \quad \|f_0\|_{L^1} \leq \|Q\|_{L^1} + \delta(\eta), \quad \|f_0\|_{L^p} \leq \|Q\|_{L^p} + \delta(\eta),$$

and let $f(t) \in L^\infty([0, +\infty), \mathcal{E}_p)$ be a weak solution to (1.1) satisfying (1.9) and (1.8), then there exists a translation shift $x(t) \in \mathbb{R}^N$ such that:

$$\forall t \geq 0, \quad |f(t, x + x(t), v) - Q|_{\mathcal{E}} < \eta.$$

A similar statement holds in the critical case $(N, \alpha) = (4, 2)$ up to an additional time dependent rescaling of the solution, see [22] for precise statements.

A different strategy to attack the question of the nonlinear stability is to consider coercivity properties of the linearized Hamiltonian as already performed in the pioneering works by Antonov, [1]. Let us for simply restrict our attention to the case of the polytropes of Proposition 1.1. Consider the energy-Casimir functional which is formally conserved by the flow of (1.1):

$$\mathcal{H}_C(f) = \frac{|f|_{L^p}^p}{p} + |f|_{L^1} + \mathcal{H}(f) = \int_{\mathbb{R}^{2N}} \left(\frac{|f|^p}{p} + f + \frac{|v|^\alpha}{\alpha} f \right) - \frac{1}{2} \int_{\mathbb{R}^N} |E_f|^2. \quad (1.19)$$

Then this functional is continuously differentiable on \mathcal{E} and Q is a critical point in the following sense: let

$$K = \text{Supp}(Q) = \left\{ (x, v) \in \mathbb{R}^N \times \mathbb{R}^N \quad \text{such that} \quad \frac{|v|^\alpha}{\alpha} + \phi_Q(x) + 1 \leq 0 \right\},$$

then from (1.16) and (1.19):

$$\forall f \in \mathcal{C}_0^\infty(K), \quad d\mathcal{H}_C(Q)f = \int_{\mathbb{R}^{2N}} \left(Q^{p-1} + 1 + \frac{|v|^\alpha}{\alpha} + \phi_Q \right) f = 0.$$

The Hessian on $C_0^\infty(K)$ is given by

$$d^2\mathcal{H}_C(Q)(f, f) = (p-1) \int_K Q^{p-2} f^2 - \int_{\mathbb{R}^N} |E_f|^2. \quad (1.20)$$

The understanding of the coercivity properties of this quadratic form in sufficiently strong norms will allow from a simple bootstrap argument to prove the nonlinear stability of the polytrope. Of course this approach can be generalized to any stationary solution and in particular provides a strategy to prove nonlinear stability without any variational structure. This problem has been addressed in several places in both the physics and mathematics litterature. It is known that this quadratic form will be coercive for a well chosen class of perturbations called ‘‘admissible’’ perturbations, see for example [40], [20], [32]. A similar approach has been used recently by Guo and Rein to prove conditional stability for the King type steady states of the Vlasov-Poisson system [17] and by Hadzic and Rein [36] for the relativistic gravitational Vlasov-Poisson system. However, this kind of structure requires to restrict the class of the perturbation theory, whereas the perturbations authorized in the present paper are in an open set of the energy space, which contains in particular these ‘‘admissible’’ perturbations. A different approach is developed by Wan [42] which obtains coercivity results for a large class of quadratic forms similar to (1.20), which imply the proof of the nonlinear stability of ground states for a large class of nonvariational problems. However, the specific case of the polytropes or more generally the solutions to (1.12) do not enter this theory due to their lack of \mathcal{C}^1 regularity on the boundary of their domain.

Eventually, the linearized Vlasov-Poisson system close to a large class of ground states was also considered in Batt, Morrison, Rein [3], but their analysis is restricted to stationary solutions for which the quadratic form of the linearized energy-Casimir functional is the sum of two positive terms and thus directly coercive. In particular, none of the ground states obtained from variational techniques in the energy space in for example [16] or [22] is covered by this analysis.

Let us stress onto the fact that for the polytropes which have a nice variational characterization, the sharp understanding of the coercivity properties of the quadratic form (1.20) allows a *quantification* of the orbital stability statement which is crucial for the further understanding of the properties of the flow of (1.1) close to Q . By sharp we mean a precise understanding of the instability directions. This situation is similar to the one for the nonlinear Schrödinger equation $iu_t = -\Delta u - |u|^{p-1}u$ or the Korteweg-de-Vries equation $u_t + (u_{xx} + u^p)_x = 0$. Indeed, both these Hamiltonian systems admit for a suitable range of the parameter p ground state type stationary solutions which are orbitally stable, see Cazenave, Lions [6]. For these two systems, another proof of the orbital stability has been given by Weinstein, [43], [44], by linearizing the conservation laws around the ground state and studying the coercivity properties of the obtained quadratic forms. Moreover, this work provided a preliminary investigation of the dispersive structure of the linearized operator close to the ground state. The obtained estimates are *the starting point* of a number of recent works regarding the dynamical stability of some specific solutions to these systems, see for example Martel, Merle and Tsai for the stability of the multisolitary waves for the (KdV) equation [30], or Bourgain and Wang [5], Merle and Raphaël [31], for the stability of some nonlinear blow up dynamics for the (NLS) system.

1.3 Statement of the results

Our aim in this paper is to adapt for (1.1) Weinstein's analysis in [43] which is the starting point for the further investigation of the nonlinear dynamics of (1.1). In a forthcoming work [24], we will in particular prove the existence and the stability of self-similar solutions for the three dimensional relativistic Vlasov-Poisson system (1.4), and the proof will partly rely on the understanding of the linearized operator close to the ground state as studied in this paper. More generally, our aim is like for the (NLS) system to be able to *quantify* the orbital stability statement of Theorem 1.2, and the obtained estimates are one of the keys to further understand the *dynamical* couplings induced by the flow near Q .

Let us consider the quadratic form (1.20) obtained by linearizing the energy-Casimir functional near the polytrope Q :

$$d^2\mathcal{H}_C(Q)(f, f) = (p-1) \int_K Q^{p-2} f^2 - \int_{\mathbb{R}^N} |E_f|^2.$$

Even though this quadratic form is not positive on its domain, we claim that we can deduce from the variational structure of Q given by Proposition 1.1 the sharp coercive structure of this quadratic form. More precisely, let us denote

$$(f, g) = \int_K fg \, dx dv$$

the $L^2(K, dx dv)$ scalar product and consider on K the weighted L^2 measure associated with Q :

$$d\mu = Q^{p-2} dx dv.$$

For $f \in L^2(K, d\mu)$, we introduce the linear operator

$$\mathcal{M}f = ((p-1)Q^{p-2}f + \phi_f) \mathbf{1}_K, \quad (1.21)$$

related to the quadratic form

$$(\mathcal{M}f, f) = (p-1) \int_K Q^{p-2} f^2 - \int_{\mathbb{R}^N} |E_f|^2, \quad (1.22)$$

and claim:

Theorem 1.3 (Coercivity of the linearized energy-Casimir functional) *Let (N, α) satisfying (1.2) and $p_{crit} < p < +\infty$. Then the quadratic form $(\mathcal{M}f, f)$ defined by (1.22) is continuous and self-adjoint on $L^2(K, d\mu)$ and there exists a universal constant $\delta = \delta(N, \alpha, p) > 0$ such that for all $f \in L^2(K, d\mu)$, we have:*

(i) if $N \neq \alpha + 2$,

$$(\mathcal{M}f, f) \geq \delta \int_K f^2 Q^{p-2} dx dv - \frac{1}{\delta} \left\{ \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right)^2 + \sum_{i=1}^N (f, x_i)^2 \right\};$$

(ii) if $N = \alpha + 2$,

$$(\mathcal{M}f, f) \geq \delta \int_K f^2 Q^{p-2} dx dv - \frac{1}{\delta} \left\{ \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right)^2 + \sum_{i=1}^N (f, x_i)^2 + (f, |v|^{2-\alpha} |x|^2)^2 \right\}.$$

Following [44], Theorem 1.3 provides a *quantitative* proof of the orbital stability of the ground state Q . Let us stress again the fact that this improvement is one the key ingredients of the *nonlinear dynamical* analysis of the three dimensional relativistic Vlasov-Poisson system in the forthcoming work [24].

The quadratic form $(\mathcal{M}f, f)$ is intimately related to the linearized Vlasov-Poisson system which is obtained by linearizing (1.1) around Q :

$$(LVP) \quad \begin{cases} \partial_t f + \mathcal{L}f = 0, \\ f(t=0, x, v) = f_0(x, v), \end{cases} \quad (1.23)$$

with

$$\mathcal{L}f = |v|^{\alpha-2} v \cdot \nabla_x f - E_Q \cdot \nabla_v f - E_f \cdot \nabla_v Q. \quad (1.24)$$

For a specific set of initial data, the linearized energy-Casimir functional $(\mathcal{M}f, f)$ is conserved by the flow of (1.24), and this allows us to prove that the linearized system (1.23) displays at most algebraic instabilities. More precisely, consider the space:

$$\mathcal{L}\mathcal{E} = \{f \in L^1_{loc}(\mathbb{R}^{2N}) \text{ with } f \mathbf{1}_K \in L^2(K, d\mu) \text{ and } f \mathbf{1}_{K^c} \in \mathcal{E}\}, \quad (1.25)$$

where $K^c = \mathbb{R}^{2N} \setminus K$, then:

Theorem 1.4 (Algebraic instability for the linearized equations) *Let $(N, \alpha) \in \{(3, 2), (4, 2)\}$ and $f_0 \in \mathcal{L}\mathcal{E}$. Then (1.23) admits a unique solution $f(t) = e^{-t\mathcal{L}}f_0 \in C(\mathbb{R}_+, \mathcal{L}\mathcal{E})$. Moreover, we have the following estimates:*

(i) *General dynamics: There holds the growth estimate:*

$$\forall t \in \mathbb{R}_+, \quad \left| e^{-t\mathcal{L}}f_0 \right|_{\mathcal{L}\mathcal{E}} \leq C(1+t^k)|f_0|_{\mathcal{L}\mathcal{E}}, \quad (1.26)$$

with $k = 2$ for $N = 3$, $k = 3$ for $N = 4$.

(ii) *Dynamics on K : There holds a decomposition $L^2(K, d\mu) = M \oplus S$ where the spaces M , S are both invariant through the flow $e^{-t\mathcal{L}}$ and S is finite dimensional. Moreover, we have:*

$$\forall g_0 \in M, \quad \left| e^{-t\mathcal{L}}g_0 \right|_{L^2(K, d\mu)} \leq C|g_0|_{L^2(K, d\mu)}, \quad (1.27)$$

$$\forall g_0 \in S, \quad \left| e^{-t\mathcal{L}}g_0 \right|_{L^2(K, d\mu)} \leq \begin{cases} C(1+t)|g_0|_{L^2(K, d\mu)} & \text{for } N = 3, \\ C(1+t^2)|g_0|_{L^2(K, d\mu)} & \text{for } N = 4. \end{cases}$$

In fact, we have a complete understanding of the dispersive properties of the flow $e^{-t\mathcal{L}}$. On the support of K , the decomposition $L^2(K, d\mu) = M \oplus S$ is explicit, see Lemma 3.6. S is the so called finite dimensional ‘‘flag’’ space which contains the algebraic instabilities generated by the large group of symmetries (1.7). On the contrary, the linear dynamics are bounded on M according to (1.27). Note that *no further dispersion holds* due to the fact that the quadratic form $(\mathcal{M}f, f)$ is conserved by the flow for $\text{Supp}(f) \subset K$ and K is a compact set.

Let now $\text{Supp}(f_0) \subset K^c$, then the solution decomposes into a part supported on K and a part supported outside K . For this last part, the flow (1.23) reduces exactly to the linear transport by the gravitational field E_Q which is explicit. Note that the characteristic curves of this field are contained in the level sets of $e(x, v) = \frac{|v|^\alpha}{\alpha} + \phi_Q$ and are trapped for $e < 0$, hence no dispersion occurs again, and non trapped for $e \geq 0$, hence an explicit linear dispersion holds. The part supported on K is proved to grow at most algebraically thanks to a Gronwall type argument, and this concludes the proof of Theorem 1.4.

Remark 1.5 *We have focused in this paper onto the polytropic ground states only. Let us recall that the class of ground states solutions is much wider and a large set of convex functions j is known to generate a ground state $Q(j)$, see [16], [22]. If we aim at treating the case of a minimizer obtained from (1.12) for a more general convex function j , a classical difficulty will occur which is the understanding of the kernel of the linearized operator. For $j(f) = f^p$, this kernel is explicit, see Lemma 2.4. A similar statement is unknown for general j . Note that similar issues are in fact addressed in Wan [42].*

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2 Coercivity of the linearized energy-Casimir functional

This section is devoted to the proof of Theorem 1.3. We shall adapt to our setting the analysis by Weinstein [43]. The proof relies on two main ingredients which are the variational characterization of the ground state as given by Proposition 1.1, and the complete description

of the kernel of \mathcal{M} . This last fact relies in part on the uniqueness of the ground state Q which is typically a delicate problem for (NLS) type of equations, see Weinstein [43], Kwong [21] and Maris [29], but is simple in our case thanks to the scaling invariance of (1.17).

2.1 The linearized problem for the potential

In this section, we study the linearized problem around ϕ_Q of the nonlinear elliptic equation (1.17). We will in particular give an explicit description of the kernel of the corresponding Schrödinger operator that implies the explicit description of the kernel of \mathcal{M} .

The nonlinear elliptic equation (1.17) linearized around ϕ_Q is

$$\mathcal{A}\phi = 0 \quad \text{with} \quad \mathcal{A} = -\Delta - V_Q$$

and with

$$V_Q(x) = \gamma_{\alpha,p,N} \left(\frac{N}{\alpha} + \frac{1}{p-1} \right) (-1 - \phi_Q(x))_+^{\frac{N}{\alpha} - \frac{p-2}{p-1}} = \frac{1}{p-1} \int_{\frac{|v|^\alpha}{\alpha} < -\phi_Q(x)-1} \frac{dv}{Q^{p-2}(x,v)}, \quad (2.1)$$

where $\gamma_{\alpha,p,N}$ was defined by (1.18). Note that $\frac{N}{\alpha} - \frac{p-2}{p-1} > 0$ under (1.2) and thus V_Q is a continuous function with compact support on \mathbb{R}^N . Hence, classical operator theory (see e.g. Reed and Simon [34] vol. 4, Theorems XIII.15 and XIII.12) gives that the operator \mathcal{A} on $L^2(\mathbb{R}^N)$ with the domain $H^2(\mathbb{R}^N)$ is self-adjoint, and that its spectrum can be written

$$\sigma(\mathcal{A}) = \{\lambda_i < 0\}_{1 \leq i \leq I} \cup [0, +\infty), \quad (2.2)$$

where $\{\lambda_i < 0\}_{1 \leq i \leq I}$ is the finite set of nonpositive eigenvalues with finite multiplicity and $[0, +\infty)$ is the essential spectrum of \mathcal{A} . 0 may be an eigenvalue. We shall denote by $(\psi_j)_{1 \leq j \leq J}$, $|\psi_j|_{L^2} = 1$, the finite set of eigenvectors associated to the nonpositive eigenvalues that are well localized in space from standard argument.

Let \dot{H}^1 be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $|u|_{\dot{H}^1} = |\nabla u|_{L^2}$, or equivalently:

$$\dot{H}^1 = \left\{ \phi \in L_{loc}^1(\mathbb{R}^N) : \frac{\phi}{\sqrt{1+|x|^2}} \in L^2(\mathbb{R}^N) \text{ and } \nabla \phi \in L^2(\mathbb{R}^N) \right\}.$$

We have the following coercivity property:

Lemma 2.1 (Coercivity of the linearized problem close to ϕ_Q) *Let (α, N, p) be as in Theorem 1.3. Then the set of functions $\phi \in \dot{H}^1$ such that $\mathcal{A}\phi = 0$ in the distributional sense coincides with the kernel of \mathcal{A} which can be characterized as*

$$\text{Ker}(\mathcal{A}) = \text{span}\{\partial_{x_i}\phi_Q\}_{1 \leq i \leq N}. \quad (2.3)$$

Moreover, there exists $c_0 > 0$ such that for all $\phi \in \dot{H}^1$ with radial symmetry,

$$\int_{\mathbb{R}^N} |\nabla \phi|^2 - \int_{\mathbb{R}^N} V_Q |\phi|^2 \geq c_0 \int_{\mathbb{R}^N} |\nabla \phi|^2 - \frac{1}{c_0} \sum_{j=1}^J (\phi, \psi_j)^2. \quad (2.4)$$

Proof of Lemma 2.1.

We follow Weinstein's strategy [43], proof of Proposition 2.8b, see also Maris [29].

Step 1. Decomposition into spherical harmonics.

Let $\phi \in \text{Ker}(\mathcal{A})$, then:

$$-\Delta\phi = V_Q\phi \quad \text{with} \quad \phi \in \dot{H}^1 \hookrightarrow L^{\frac{2N}{N-2}}, \quad (2.5)$$

hence $\phi \in C^2(\mathbb{R}^N)$ from standard elliptic theory. One can thus decompose ϕ into spherical harmonics. More precisely, let \mathcal{P}_k be the space of spherical harmonics of degree k , with $\dim \mathcal{P}_k = a_k = C_{N+k-1}^k - C_{N+k-3}^k$, and for each k , let $\{Y_i^k\}_{1 \leq i \leq a_k}$ be the L^2 orthonormal basis of \mathcal{P}_k . Then ϕ has a unique expansion

$$\phi(x) = \sum_{k=0}^{+\infty} \sum_{i=1}^{a_k} \varphi_{k,i}(|x|) Y_i^k \left(\frac{x}{|x|} \right), \quad (2.6)$$

with

$$\varphi_{k,i}(|x|) = \int_{S^{N-1}} \phi(|x|\theta) Y_i^k(\theta) d\theta \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty. \quad (2.7)$$

The potential V_Q having radial symmetry, (2.5) implies

$$\mathcal{A}_k \varphi_{k,i} = 0, \quad \text{with} \quad \mathcal{A}_k = -\frac{d^2}{dr^2} - \frac{N-1}{r} \frac{d}{dr} + \frac{k(k+N-2)}{r^2} - V_Q(r). \quad (2.8)$$

Let $r_Q > 0$ be the unique solution to

$$\phi_Q(r_Q) = -1, \quad (2.9)$$

then the potential V_Q is compactly supported on $[0, r_Q]$. Hence one can solve (2.8) explicitly outside its support with the constraint $\varphi_{k,i} \in L^\infty(\mathbb{R})$ deduced from $\phi \in L^\infty(\mathbb{R}^N)$ and (2.6). We get:

$$\forall k \geq 0 \quad \forall i \in \{1, \dots, a_k\} \quad \forall r > r_Q \quad \varphi_{k,i}(r) = \frac{C_{k,i}}{r^{k+N-2}} \quad (2.10)$$

for some constant $C_{k,i}$.

Observe now that (2.3) is equivalent to:

$$\varphi_{k,i} = 0 \quad \text{when} \quad k \neq 1 \quad \text{and} \quad \varphi_{1,i}(r) = a_{1,i} \phi'_Q(r) \quad \text{for} \quad 1 \leq i \leq N. \quad (2.11)$$

Step 2. The case $k \geq 1$.

Let $k \geq 1$. Observe that (2.10) implies $\phi_{k,i} \in H_r^1$ where H_r^1 denotes the set of H^1 distributions of \mathbb{R}^N with radial symmetry. We now take the derivative of (1.17) with respect to the radial coordinate r and get after direct calculations:

$$\mathcal{A}_1 \phi'_Q = 0.$$

Therefore ϕ'_Q is an eigenfunction of \mathcal{A}_1 corresponding to the eigenvalue zero. Observe from (1.17) that ϕ'_Q is nonnegative on $(0, +\infty)$ and it follows from standard spectral analysis [34] that it is the ground state of \mathcal{A}_1 on H_r^1 . We conclude that:

$$\forall w \in H_r^1 \quad (\mathcal{A}_1 w, w) \geq 0 \quad \text{and} \quad (\text{Ker} \mathcal{A}_1)_{H_r^1} = \text{span}(\phi'_Q), \quad (2.12)$$

and the case $k = 1$ of (2.11) follows.

For $k \geq 2$, we have from (2.8):

$$0 = (\mathcal{A}_k \phi_{k,i}, \phi_{k,i}) = (\mathcal{A}_1 \phi_{k,i}, \phi_{k,i}) + (k(k + N - 2) - (N - 1)) \int_0^{+\infty} |\phi_{k,i}|^2 r^{N-3} dr,$$

which gives $\phi_{k,i} = 0$ thanks to the positivity of \mathcal{A}_1 (2.12), and the case $k \geq 2$ in (2.10) is also solved.

Step 3. The case $k = 0$.

It remains the case $k = 0$ that has to be treated in a different way. The fact that $\varphi_0 = 0$ is a consequence of the scaling structure of the ϕ_Q equation (1.17). Indeed, φ_0 solves

$$\mathcal{A}_0 \varphi_0 = \left(-\frac{d^2}{dr^2} - \frac{N-1}{r} \frac{d}{dr} - V_Q \right) \varphi_0 = 0 \quad (2.13)$$

and, by (2.10), satisfies $\varphi_0(r) = \frac{C_0}{r^{N-2}}$ for r large enough. In particular,

$$\varphi_0(r) \rightarrow 0 \quad \text{as } r \rightarrow +\infty. \quad (2.14)$$

Remark also that from the C^2 regularity of ϕ that $\varphi_0'(0) = 0$.

Let now:

$$h = -1 - \phi_Q \quad \text{and} \quad \beta = \frac{2\alpha(p-1)}{\alpha + (N-\alpha)(p-1)}, \quad (2.15)$$

we claim that:

$$\mathcal{A}_0 H = 0 \quad \text{with } H(r) = \beta h(r) + r h'(r). \quad (2.16)$$

Indeed, for $\lambda > 0$, let $h_\lambda(r) = \lambda^\beta h(\lambda r)$. From (1.17) and the above choice of β , h_λ solves

$$\left(-\frac{d^2}{dr^2} - \frac{N-1}{r} \frac{d}{dr} \right) h_\lambda - \gamma_{\alpha,p,N} (h_\lambda)_+^{\frac{1}{p-1} + \frac{N}{\alpha}} = 0.$$

Differentiating this expression with respect to λ and evaluating the result at $\lambda = 1$ yields (2.16). We now observe from (2.10), (2.15) and (2.16) that $H(r) \rightarrow -\beta \neq 0$ as $r \rightarrow +\infty$. From standard ODE analysis, all the solutions to (2.13) with a vanishing derivative at $r = 0$ are proportional. Since $H'(0) = -(\beta + 1)\phi_Q'(0) = 0$, both functions φ_0 and H are proportional, which implies by (2.14) that φ_0 is identically zero and concludes the proof of (2.11). The proof of (2.3) is now complete.

Step 4. Proof of (2.4).

We now conclude the proof of (2.4) which follows from standard variational arguments. We briefly sketch the proof for the sake of completeness. Let the quadratic form

$$(\mathcal{A}\phi, \phi) = \int |\nabla \phi|^2 dx - \int V_Q |\phi|^2 dx$$

that is continuous on \dot{H}^1 since V_Q is compactly supported. Let Λ be the set of $\phi \in \dot{H}^1$ with radial symmetry such that

$$(\phi, \psi_j) = 0 \quad \text{for } j = 1, \dots, J. \quad (2.17)$$

Note that these L^2 scalar products are well defined since the ψ_k 's are well localized in space. The spectral property (2.2) implies:

$$\forall \phi \in \Lambda \cap L^2(\mathbb{R}^N) \quad (\mathcal{A}\phi, \phi) \geq 0. \quad (2.18)$$

From a standard density argument, (2.18) holds also on Λ . We claim that, in fact,

$$\inf_{\phi \in \Lambda, |\nabla \phi|_{L^2} = 1} (\mathcal{A}\phi, \phi) > 0. \quad (2.19)$$

This together with the continuity of the quadratic form $(\mathcal{A}\phi, \phi)$ on \dot{H}^1 now implies (2.4).

Proof of (2.19): We argue by contradiction and consider a sequence ϕ_n such that

$$\phi_n \in \Lambda, \quad (\mathcal{A}\phi_n, \phi_n) \rightarrow 0 \quad \text{and} \quad |\nabla \phi_n|_{L^2} = 1. \quad (2.20)$$

Up to a subsequence, $\phi_n \rightharpoonup \phi$ in \dot{H}_r^1 . Moreover, the Sobolev embedding $\dot{H}^1 \hookrightarrow L^{\frac{2N}{N-2}}$ being locally compact, we have $\phi \in \Lambda$ and:

$$1 - (\mathcal{A}\phi_n, \phi_n) = \int V_Q |\phi_n|^2 \rightarrow \int V_Q |\phi|^2 = 1 \quad \text{as } n \rightarrow +\infty.$$

By lower semicontinuity, $|\nabla \phi|_{L^2} \leq 1$ and thus $(\mathcal{A}\phi, \phi) \leq 0$. Since $\phi \in \Lambda$, this implies

$$(\mathcal{A}\phi, \phi) = 0, \quad |\nabla \phi|_{L^2} = 1 \quad (2.21)$$

and the convergence $\phi_n \rightarrow \phi$ holds in the strong \dot{H}^1 topology. Hence $\inf(\mathcal{A}\phi, \phi)$ is attained and the Euler-Lagrange equation of this constrained variational problem reads

$$-\lambda \Delta \phi - V_Q \phi = \sum_{j=1}^J b_j \psi_j. \quad (2.22)$$

We take the L^2 inner product of (2.22) with ϕ and get $\lambda |\nabla \phi|_{L^2}^2 = \int V_Q |\phi|^2 = 1$ thanks to the orthogonality conditions (2.20). Thus $\lambda = 1$ and (2.22) becomes:

$$\mathcal{A}\phi = \sum_{j=1}^J b_j \psi_j. \quad (2.23)$$

Taking the scalar product of (2.23) with ψ_{j_0} gives now

$$b_{j_0} = (\mathcal{A}\phi, \psi_{j_0}) = \lambda_{j_0}(\phi, \psi_{j_0}) = 0$$

where we also used (2.17) again. Hence $b_{j_0} = 0$ and $\phi \in \text{Ker}(\mathcal{A})$ from (2.23). It remains to remark that (2.3) and the radial symmetry of ϕ imply that $\phi = 0$, which contradicts (2.21). The proof of (2.19) is complete.

This concludes the proof of Lemma 2.1.

2.2 Variational estimates and proof of Theorem 1.3

In this subsection, we study the linear operator \mathcal{M} defined by (1.21) on $L^2(K, d\mu)$ and prove Theorem 1.3.

Let us start with the following continuity result:

Lemma 2.2 (Continuity of \mathcal{M} on $L^2(K, d\mu)$) *Let (α, N, p) be as in Theorem 1.3. Then the quadratic form $(\mathcal{M}f, f)$ is continuous and self-adjoint on $L^2(K, d\mu)$. Moreover, let a sequence $f_n \in L^2(K, d\mu)$ be such that*

$$f_n \rightharpoonup f \quad \text{in } L^2(K, d\mu), \quad (2.24)$$

then

$$E_{f_n} \rightarrow E_f \quad \text{in } L^2(\mathbb{R}^N). \quad (2.25)$$

Proof of Lemma 2.2.

Let the potential V_Q given by (2.1), then from Cauchy-Schwarz: $\forall f \in L^2(K, d\mu)$,

$$|\rho_f(x)| = \left| \int_K f(x, v) dv \right| \leq \left((p-1) \int_K f^2(x, v) Q^{p-2} dv \right)^{1/2} (V_Q(x))^{1/2}. \quad (2.26)$$

Observe now that the potential V_Q is a continuous function with compact support on \mathbb{R}^N . Thus (2.26) implies $\rho_f \in L^1 \cap L^2(\mathbb{R}^N)$ with $\text{Supp}(\rho_f) \subset \{|x| \leq r_Q\}$ (r_Q was defined by (2.9)). Sobolev embeddings now imply $E_f \in L^2(\mathbb{R}^N)$ and the continuity of \mathcal{M} on $L^2(K, d\mu)$ follows. The fact that \mathcal{M} is self-adjoint follows from integration by parts.

Let now a sequence f_n satisfying (2.24). The estimate (2.26) gives an $L^1 \cap L^2$ bound for ρ_{f_n} , and thus E_{f_n} is locally compact in $L^2(\mathbb{R}^N)$ from Sobolev embeddings. Observe now that $|x| > 2r_Q$ and $|x - y| < r_Q$ imply $|y| > |x| - r_Q > \frac{|x|}{2}$ and thus:

$$|E_f(x)| \leq C \int_{|x-y| \leq r_Q} \frac{|\rho_f(x-y)|}{|y|^{N-1}} dy \leq C \frac{2^{N-1}}{|x|^{N-1}} |\rho_f|_{L^1}.$$

We conclude that E_{f_n} is L^2 compact and (2.25) follows. This concludes the proof of Lemma 2.2.

We now claim the following positivity property for \mathcal{M} that is a consequence of the variational characterization of Q as given by Proposition 1.1 and is the very heart of the proof of Theorem 1.3.

Lemma 2.3 (Positivity of \mathcal{M} induced by the variational structure of Q) *Let (α, N, p) be as in Theorem 1.3. Let $f \in L^2(K, d\mu)$ with*

$$\left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) = 0. \quad (2.27)$$

Then the quadratic form defined by (1.22) satisfies

$$(\mathcal{M}f, f) \geq 0. \quad (2.28)$$

Proof of Lemma 2.3.

Let $f \in C_0^\infty(K)$. Then for any η small enough, $Q + \eta f \in \mathcal{E}$. Let $J_{\alpha,p,N}$ be the functional defined by (1.14) and denote $J(\eta) = J_{\alpha,p,N}(Q + \eta f)$. The variational characterization of Q given by Proposition 1.1 implies

$$J'(0) = 0, \quad J''(0) \geq 0. \quad (2.29)$$

Now, from a direct computation using the identities

$$\omega := \frac{\alpha\theta_1}{\| |v|^\alpha Q \|_{L^1}} = \frac{\theta_2}{\| Q \|_{L^p}^p} = \frac{\theta_3}{\| Q \|_{L^1}} = \frac{2}{\| E_Q \|_{L^2}^2}$$

obtained during the construction of Q (see [22]), we get

$$\begin{aligned} (\mathcal{M}f, f) &= \frac{J''(0)}{\omega J(0)} + \omega \left\{ \frac{\alpha + 2 - N}{N - 2} \left(f, \frac{|v|^\alpha}{\alpha} \right)^2 + \frac{p}{\theta_2} (f, Q^{p-1})^2 + \frac{1}{\theta_3} (f, 1)^2 \right\} \\ &\quad + \omega \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) \left(f, \frac{|v|^\alpha}{\alpha} - \phi_Q \right). \end{aligned}$$

Note that $3 \leq N \leq \alpha + 2$ in the range of parameters (1.2), and thus (2.29) and the orthogonality condition (2.27) now imply (2.28). The general case $f \in L^2(K, d\mu)$ follows by density. This concludes the proof of Lemma 2.3.

The second key to the proof of Theorem 1.3 is the fact that kernel of \mathcal{M} is explicit and in particular \mathcal{M} is invertible when restricted to radially symmetric distributions. Moreover, some inverses are explicit as a consequence of the action of the large group of symmetries (1.7).

Lemma 2.4 (Explicit description of the the kernel of \mathcal{M}) *Let (α, N, p) be as in Theorem 1.3. Then:*

$$\text{Ker}(\mathcal{M}) = \{ f \in L^2(K, d\mu) \text{ with } \mathcal{M}f = 0 \} = \text{span}\{ \partial_{x_i} Q \}_{1 \leq i \leq N}. \quad (2.30)$$

Moreover, let

$$S_1 = \frac{N - \alpha}{2} x \cdot \nabla_x Q - v \cdot \nabla_v Q, \quad S_2 = -x \cdot \nabla_v Q, \quad S_3 = \mathcal{M}^{-1} \left(\frac{|v|^{2-\alpha} |x|^2}{2} \right), \quad (2.31)$$

then we have the following identities:

$$\mathcal{M}S_1 = \alpha \left(\frac{|v|^\alpha}{\alpha} + \phi_Q \right) \mathbf{1}_K, \quad \mathcal{M}S_2 = x \cdot v |v|^{\alpha-2} \mathbf{1}_K, \quad \mathcal{M}S_3 = \frac{|v|^{2-\alpha} |x|^2}{2} \mathbf{1}_K. \quad (2.32)$$

Proof of Lemma 2.4.

Step 1. Description of $\text{Ker}(\mathcal{M})$.

We claim that:

$$\text{span} \{ \partial_{x_i} Q \}_{1 \leq i \leq N} \subset \text{Ker}(\mathcal{M}).$$

Indeed, we rewrite the equation (1.16) of Q :

$$\left(Q^{p-1} + \frac{|v|^\alpha}{\alpha} + \phi_Q + 1 \right) \mathbf{1}_K = 0 \quad (2.33)$$

and take a derivative with respect to $(x_i)_{1 \leq i \leq N}$ to derive:

$$((p-1)Q^{p-2}\partial_{x_i}Q + \partial_{x_i}\phi_Q) \mathbf{1}_K = 0. \quad (2.34)$$

In particular,

$$\int_{\mathbb{R}^{2N}} Q^{p-2} (\partial_{x_i}Q)^2 = \frac{1}{p-1} \int_{\mathbb{R}^N} (\partial_{x_i}\phi_Q)^2 V_Q < +\infty,$$

where V_Q is the potential defined by (2.1), and thus $\partial_{x_i}Q \in L^2(K, d\mu)$. Using also $\partial_{x_i}\phi_Q = \phi_{\partial_{x_i}Q}$, we deduce from (2.34) that $\partial_{x_i}Q \in \text{Ker}(\mathcal{M})$.

Let now $f \in \text{Ker}(\mathcal{M})$. For $(x, v) \in K$ we have

$$(p-1)Q^{p-2}f(x, v) + \phi_f(x) = 0 \quad (2.35)$$

and thus, for all $|x| \leq r_Q$,

$$\Delta\phi_f(x) = \rho_f(x) = -\frac{1}{p-1} \int_K \frac{\phi_f(x)}{Q^{p-2}(x, v)} dv = -\phi_f(x) V_Q(x). \quad (2.36)$$

For $|x| > r_Q$, $\rho_f(x) = 0$, so (2.36) still holds. We conclude that ϕ_f belongs to the kernel of the operator $\mathcal{A} = -\Delta - V_Q$ on \dot{H}^1 . By Lemma 2.1, there exists $\{c_i\}_{1 \leq i \leq N}$ such that

$$\phi_f = \sum_{i=1}^N c_i \partial_{x_i}\phi_Q.$$

It follows from (2.34) and (2.35) that

$$f = \sum_{i=1}^N c_i \partial_{x_i}Q$$

and this concludes the proof of (2.30).

Step 2. Derivation of the algebraic identities (2.32).

The first identity in (2.32) is a consequence of the scaling invariance of (2.33). For a parameter $\mu > 0$, define

$$Q_\mu(x, v) = \mu^{N-\alpha} Q(x, \mu v),$$

then the corresponding microscopic energy defined by

$$e(x, v) = \frac{|v|^\alpha}{\alpha} + \phi_Q(x)$$

scales according to $e_\mu(x, v) = \frac{1}{\mu^\alpha} e(x, \mu v)$. We thus compute from (2.33):

$$\left(\mu^{-(p-1)(N-\alpha)} Q_\mu^{p-1} + \mu^\alpha e_\mu(x, v) + 1 \right) \mathbf{1}_{(x, \mu v) \in K} = 0. \quad (2.37)$$

Differentiating this relation with respect to μ and evaluating the result at $\mu = 1$ yields

$$(p-1)Q^{p-2}R - (p-1)(N-\alpha)Q^{p-1} + |v|^\alpha = 0 \quad \text{on } K,$$

where we have denoted

$$R = (N-\alpha)Q + v \cdot \nabla_v Q = \frac{dQ_\mu}{d\mu} \Big|_{\mu=1}.$$

Remarking that

$$\phi_R = -\alpha\phi_Q \quad \text{from} \quad \rho_R = \int ((N-\alpha)Q + v \cdot \nabla_v Q) dv = -\alpha\rho_Q,$$

we get the following intermediate identity:

$$\mathcal{M}R = (p-1)(N-\alpha)Q^{p-1} - \alpha \left(\frac{|v|^\alpha}{\alpha} + \phi_Q \right). \quad (2.38)$$

We now rescale the x variable in Q and set for $\lambda > 0$:

$$Q_\lambda(x, v) = \frac{1}{\lambda^2} Q\left(\frac{x}{\lambda}, v\right).$$

The microscopic energy scales according to $e_\lambda(x, v) = e\left(\frac{x}{\lambda}, v\right)$ and (2.33) becomes

$$\left(\lambda^{2(p-1)} Q_\lambda^{p-1} + e_\lambda(x, v) + 1 \right) \mathbf{1}_{(x/\lambda, v) \in K} = 0.$$

We differentiate this expression with respect to λ and evaluate the result at $\lambda = 1$ to get:

$$-(p-1)Q^{p-2}\tilde{R} + 2(p-1)Q^{p-1} - x \cdot \nabla_x \phi_Q = 0 \quad \text{on } K,$$

where we have denoted

$$\tilde{R} = 2Q + x \cdot \nabla_x Q = -\frac{dQ_\lambda}{d\lambda} \Big|_{\lambda=1}.$$

Now remarking that

$$\phi_{\tilde{R}} = x \cdot \nabla_x \phi_Q,$$

we get the second intermediate inequality:

$$\mathcal{M}\tilde{R} = 2(p-1)Q^{p-1}. \quad (2.39)$$

Multiplying (2.39) by $\frac{N-\alpha}{2}$ and subtracting (2.38) yields the first identity in (2.32).

The second identity in (2.32) can be proved by a direct computation, remarking simply that

$$\phi_{S_2} = 0 \quad \text{from} \quad \rho_{S_2} = -\int x \cdot \nabla_v Q dv = 0$$

and that by (1.16)

$$\nabla_v Q = -\frac{1}{p-1} \frac{v|v|^{\alpha-2}}{Q^{p-2}}. \quad (2.40)$$

Finally, the last identity in (2.32) is obvious as soon as we are able to define S_3 according to (2.31). To this aim, we recall that since Q is radially symmetric, we have

$\int_{\mathbb{R}^{2N}} |v|^{2-\alpha} |x|^2 \partial_{x_i} Q = 0$ for $1 \leq i \leq N$, and (2.30) implies that $\frac{|v|^{2-\alpha} |x|^2}{2} \in \text{Ker}(\mathcal{M})^\perp$. Hence Lemma 2.3 and the Lax-Milgram Theorem ensure the invertibility of \mathcal{M} on $\text{Ker}(\mathcal{M})^\perp$ and (2.31) defines S_3 in $L^2(K, d\mu)$ without ambiguity.

This concludes the proof of Lemma 2.4.

We are now in position to conclude the proof of Theorem 1.3 which follows from standard variational techniques.

Proof of Theorem 1.3.

Let \mathcal{J} be the set of $f \in L^2(K, d\mu)$ with

$$\begin{cases} \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) = (f, x_i) = 0, & 1 \leq i \leq N, & \text{if } N - \alpha \neq 2, \\ \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) = (f, x_i) = (f, |v|^{2-\alpha} |x|^2) = 0, & 1 \leq i \leq N, & \text{if } N - \alpha = 2. \end{cases} \quad (2.41)$$

Note that the L^2 inner products are well defined as for all $f \in L^2(K, d\mu)$ and $g \in L^\infty(K)$:

$$\begin{aligned} |(f, g)| &\leq |g|_{L^\infty} \left(\int_K f^2 Q^{p-2} dx dv \right)^{1/2} \left(\int_K \frac{1}{Q^{p-2}} dx dv \right)^{1/2} \\ &\leq C |g|_{L^\infty} |f|_{L^2(K, d\mu)} |V_Q|_{L^\infty}^{1/2}, \end{aligned}$$

with V_Q defined by (2.1). We now claim:

$$I = \inf_{f \in \mathcal{J}, |f|_{L^2(K, d\mu)} = 1} (\mathcal{M}f, f) > 0. \quad (2.42)$$

Since $(\mathcal{M}\cdot, \cdot)$ is a continuous quadratic form, (2.42) and the definition (2.41) of \mathcal{J} imply the coercivity property of Theorem 1.3.

Proof of (2.42). Arguing by contradiction and using the positivity property of Lemma 2.3, we let a sequence f_n such that

$$f_n \in \mathcal{J}, \quad \int_K f_n^2 Q^{p-2} = 1 \quad \text{and} \quad (\mathcal{M}f_n, f_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (2.43)$$

Up to a subsequence, $f_n \rightharpoonup f$ in $L^2(K, d\mu)$ and thus $f \in \mathcal{J}$ with:

$$\int_K f^2 Q^{p-2} dx dv \leq 1. \quad (2.44)$$

Now by Lemma 2.2, $E_{f_n} \rightarrow E_f$ in $L^2(\mathbb{R}^N)$. Since, by (2.43), we have $|E_{f_n}|_{L^2}^2 \rightarrow p-1$, we deduce that $|E_f|_{L^2}^2 = p-1$ and thus $f \neq 0$. Moreover, (2.44) implies $(\mathcal{M}f, f) \leq 0$ and thus $(\mathcal{M}f, f) = 0$ from Lemma 2.3 and $f \in \mathcal{J}$. This implies $\int_K f^2 Q^{p-2} = 1$ and the infimum I defined by (2.42) is attained at f .

We now write down the Euler-Lagrange equation for this constrained minimization problem and get the existence of Lagrange multipliers $\beta, (\gamma_i)_{1 \leq i \leq N}, \kappa, \tau$ such that

$$\mathcal{M}f = \beta Q^{p-2} f + \sum_{i=1}^N \gamma_i x_i \mathbf{1}_K + \kappa \left(\frac{|v|^\alpha}{\alpha} + \phi_Q \right) \mathbf{1}_K \quad \text{for } N - \alpha \neq 2, \quad (2.45)$$

$$\mathcal{M}f = \beta Q^{p-2}f + \sum_{i=1}^N \gamma_i x_i \mathbf{1}_K + \kappa \left(\frac{|v|^\alpha}{\alpha} + \phi_Q \right) \mathbf{1}_K + \tau |v|^{2-\alpha} |x|^2 \mathbf{1}_K \quad \text{for } N - \alpha = 2. \quad (2.46)$$

Take the $L^2(K, dx dv)$ inner product of (2.45) or (2.46) with f , use $(\mathcal{M}f, f) = 0$, the orthogonality conditions (2.41) and $\int f^2 Q^{p-2} dx dv = 1$ to obtain $\beta = 0$. Take then the inner product of (2.45) or (2.46) with $\partial_{x_i} Q$ for $1 \leq i \leq N$ to get

$$-\gamma_i \int Q = (\mathcal{M}f, \partial_{x_i} Q) = (f, \mathcal{M}\partial_{x_i} Q) = 0,$$

where we used (A.2) from the Appendix and (2.30). Hence the γ_i 's are all zero.

Let now $N - \alpha \neq 2$ and take the inner product of (2.45) with S_1 defined by (2.31). Using (2.32), we get

$$\kappa \left(\frac{|v|^\alpha}{\alpha} + \phi_Q, S_1 \right) = (\mathcal{M}f, S_1) = (f, \mathcal{M}S_1) = \alpha \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) = 0.$$

Since, by (A.1) given in Appendix, in the subcritical case $N - \alpha \neq 2$ the factor of κ is not zero, we deduce that $\kappa = 0$. Hence $\mathcal{M}f = 0$. From (2.30), we thus have the existence of $(c_i)_{1 \leq i \leq n}$ such that

$$f = \sum_{i=1}^N c_i \partial_{x_i} Q.$$

Multiplying this expression by x_i and integrating, we deduce from (2.41) that $c_i = 0$ for $i = 1, \dots, N$, thus $f = 0$ which is absurd.

Let $N - \alpha = 2$. Taking the inner product of (2.46) with S_1 and using (A.1), (2.32) and (2.41), we get

$$0 = (f, \mathcal{M}S_1) = (\mathcal{M}f, S_1) = \tau (|v|^{2-\alpha} |x|^2, S_1) = -\alpha\tau \int |v|^{2-\alpha} |x|^2 Q \quad \text{and thus } \tau = 0.$$

Take now the inner product of (2.46) with S_3 (defined by (2.31)). By the orthogonality condition (2.41) we have

$$\kappa \left(\frac{|v|^\alpha}{\alpha} + \phi_Q, S_3 \right) = (\mathcal{M}f, S_3) = (f, \mathcal{M}S_3) = \left(f, \frac{|v|^{2-\alpha} |x|^2}{2} \right) = 0,$$

so we deduce from (A.5) that $\kappa = 0$ and $\mathcal{M}f = 0$. The end of the proof is then identical to the case $N - \alpha \neq 2$.

This concludes the proof of Theorem 1.3.

3 The linearized Vlasov-Poisson system in dimension 3 or 4

In this section, we fix $\alpha = 2$ and take $N = 3$ or $N = 4$ and prove Theorem 1.4.

3.1 Well-posedness of the linearized Vlasov-Poisson system

We prove in this subsection the well-posedness of the linearized equation (1.23) and some conservation laws associated to this flow.

We start with a technical lemma stating a few useful properties of the $\mathcal{L}\mathcal{E}$ space defined by (1.25) that is a natural space for the study of the linearized Vlasov-Poisson problem.

Lemma 3.1 (Embedding of the \mathcal{E} space) *The space $\mathcal{L}\mathcal{E}$ is continuously embedded into $L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N) \cap L^{\frac{2N}{N+2}}(\mathbb{R}_x^N, L^1(\mathbb{R}_v^N))$. Moreover there exists a constant C such that:*

$$\forall f \in \mathcal{L}\mathcal{E}, \quad |E_f|_{L^2} \leq C|f|_{\mathcal{L}\mathcal{E}}. \quad (3.1)$$

Proof of Lemma 3.1.

Let $f \in \mathcal{L}\mathcal{E}$ and decompose this function into $f = f^i + f^e$, with $f^i = f\mathbf{1}_K \in L^2(K, d\mu)$ and $f^e = f\mathbf{1}_{K^c} \in \mathcal{E}$. By (2.26) we have

$$|f^i|_{L^1} + |f^i|_{L_x^2 L_v^1} \leq C|f^i|_{L^2(K, d\mu)},$$

and, by standard interpolation inequality,

$$|f^e|_{L^1} + |f^e|_{L_x^q L_v^1} \leq C|f^e|_{\mathcal{E}},$$

with $q = \frac{(N+2)p-N}{Np-N+2}$. The assumption $p > p_{crit}$ ensures that $q > \frac{2N}{N+2}$, thus, remarking that, also, $2 > \frac{2N}{N+2}$, we get

$$|f|_{L^1} + |f|_{L_x^{\frac{2N}{N+2}} L_v^1} \leq C|f|_{\mathcal{L}\mathcal{E}}.$$

The estimate (3.1) of the field now follows from the Poisson equation and the generalized Young inequality. This concludes the proof of Lemma 3.1.

We now state the well-posedness of the linearized Vlasov-Poisson system (1.23) that is the main result of this subsection.

Proposition 3.2 (Properties of the linearized flow in \mathcal{E}) *Let $(N, \alpha) = (3, 2)$ or $(4, 2)$. Let $f_0 \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, then (1.23) admits a unique weak solution $e^{-t\mathcal{L}}f_0 \in C(\mathbb{R}_+, L^1(\mathbb{R}^N \times \mathbb{R}^N))$. Assume moreover that $f_0 \in \mathcal{L}\mathcal{E}$ and decompose it as follows:*

$$f_0 = f_0^i + f_0^e \quad \text{with} \quad f_0^i = f_0 \mathbf{1}_K, \quad f_0^e = f_0 \mathbf{1}_{K^c}. \quad (3.2)$$

Let $f^i(t) = e^{-t\mathcal{L}}f_0^i$, $f^e(t) = e^{-t\mathcal{L}}f_0^e$. Then for all $t \geq 0$, we have $f(t) = e^{-t\mathcal{L}}f_0 = f^i(t) + f^e(t) \in \mathcal{L}\mathcal{E}$ and the following conservation laws hold: $\forall t \geq 0$,

$$\text{Supp } f^i(t) \subset K, \quad (\mathcal{M}f^i, f^i)(t) = (\mathcal{M}f_0^i, f_0^i), \quad (3.3)$$

$$\int_{K^c} \left(1 + \frac{|v|^2}{2} + \phi_Q(x, v)\right) f^e(t, x, v) dx dv = \int_{K^c} \left(1 + \frac{|v|^2}{2} + \phi_Q\right) f_0^e dx dv, \quad (3.4)$$

$$\forall q \in [1, p], \quad |f^e(t)|_{L^q(K^c)} = |f_0^e|_{L^q}, \quad (3.5)$$

$$(\mathcal{M}f^e, f^e)(t) = -2 \int_0^t \int_K f^e(s) v \cdot E_{f^e \mathbf{1}_{K^c}}(s) ds. \quad (3.6)$$

Here we extended the operator \mathcal{M} to $\mathcal{L}\mathcal{E}$ by

$$\mathcal{M}f = ((p-1)Q^{p-2}f + \phi_{f\mathbf{1}_K}) \mathbf{1}_K = \mathcal{M}(f\mathbf{1}_K). \quad (3.7)$$

Remark 3.3 From (3.3), $L^2(K, d\mu)$ is invariant under the flow (1.23) and the linearized energy-Casimir functional $(\mathcal{M}f, f)$ is conserved by the flow. This is however no longer true for a general initial data and the error to the conservation law is measured by (3.6).

Proof of Proposition 3.2

Step 1. Transport by E_Q .

Let \mathcal{T} be the linear transport operator induced by the field of the ground state Q :

$$\mathcal{T} = v \cdot \nabla_x - E_Q \cdot \nabla_v. \quad (3.8)$$

Thanks to the regularity of the field $E_Q = \nabla_x \phi_Q$, one can define the characteristics associated with \mathcal{T} . For (t, x, v) , we denote by $X(s; t, x, v)$, $V(s; t, x, v)$ the global solution of the differential system

$$\frac{d}{ds}X = V, \quad \frac{d}{ds}V = -E_Q(X), \quad X(s=t) = x, \quad V(s=t) = v. \quad (3.9)$$

Recall that the energy is an invariant of this system, i.e. $\frac{1}{2}|V(s; t, x, v)|^2 + \phi_Q(X(s; t, x, v))$ is independent of s , and that for any s, t the Jacobian of the Lagrangian change of variable

$$(x, v) \mapsto (X(s; t, x, v), V(s; t, x, v)) \quad (3.10)$$

is equal to 1. An important consequence of the energy invariant and the fact that Q is a function of the microscopic energy is that a characteristic curve cannot cross the boundary of the support Q : K and K^c are both invariant along the flow (3.9).

Step 2. Well-posedness of (1.23) in L^1 .

It is a simple consequence of the existence of the characteristics curves (3.9) and we briefly sketch the proof for the sake of completeness. Let $T > 0$ and introduce the following mapping on $\mathcal{C}([0, T], L^1(\mathbb{R}^N \times \mathbb{R}^N))$: for f in this space, $G(f)$ is defined as the unique weak solution g of

$$\partial_t g + v \cdot \nabla_x g - E_Q \cdot \nabla_v g = E_f \cdot \nabla_v Q, \quad g(t=0) = f_0, \quad (3.11)$$

given thanks to the characteristics by

$$G(f)(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)) + \int_0^t (E_f \cdot \nabla_v Q)(s, X(s; t, x, v), V(s; t, x, v)) ds. \quad (3.12)$$

It is useful to note that $f \in \mathcal{C}([0, T], L^1(\mathbb{R}^N \times \mathbb{R}^N))$ implies $\rho_f \in \mathcal{C}([0, T], L^1(\mathbb{R}^N))$ and thus $E_f \in \mathcal{C}([0, T], L^{\frac{N}{N-1}, \infty})$, where $L^{\frac{N}{N-1}, \infty}$ stands for the weak $L^{\frac{N}{N-1}}$ space (or Marcinkiewicz space). Hence $E_f \in L^1_{loc}(\mathbb{R}^N)$. Observe from (2.40) that

$$\int_{\mathbb{R}^N} |\nabla_v Q(x, v)| dv \leq CV_Q(x)$$

is bounded on \mathbb{R}^N and compactly supported. Thus the right-hand side of (3.11) belongs to $\mathcal{C}([0, T], L^1(\mathbb{R}^N \times \mathbb{R}^N))$ and for all $t \in [0, T]$:

$$|E_f \cdot \nabla_v Q|_{L^1}(t) \leq C |f|_{L^1}(t).$$

Integrating (3.12) on $\mathbb{R}^N \times \mathbb{R}^N$ and performing the Lagrangian change of coordinate (3.10), we get for any $f_1, f_2 \in C([0, T], L^1(\mathbb{R}^N \times \mathbb{R}^N))$

$$|G(f_1) - G(f_2)(t)|_{L^1} \leq C \int_0^t |f_1 - f_2|_{L^1}(s) ds.$$

This is enough to conclude by the Banach fixed point theorem for T small enough. We have proved that (1.23) admits a unique solution, that satisfies

$$f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)) + \int_0^t (E_f \cdot \nabla_v Q)(s, X(s; t, x, v), V(s; t, x, v)) ds. \quad (3.13)$$

Step 3. Well-posedness in $\mathcal{L}\mathcal{E}$.

Let now $f = f^i + f^e \in \mathcal{L}\mathcal{E}$. The simple remarks that K and K^c remain invariant under the flow of the characteristics (3.9) and that the source term $E_f \cdot \nabla_v Q$ in (1.23) is supported on K enable us to conclude that $\text{Supp } f^i \subset K$ and that $h := f^e \mathbf{1}_{K^c} = f \mathbf{1}_{K^c}$ solves in the weak sense the equation

$$\partial_t h + v \cdot \nabla_x h - E_Q \cdot \nabla_v h = 0, \quad h(t=0) = f_0^e. \quad (3.14)$$

It is clear then that $f^e \mathbf{1}_{K^c} \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N$ and that (3.4) and (3.5) hold (recall that $\frac{|v|^2}{2} + \phi_Q$ is invariant along the characteristics). Note that ϕ_Q being bounded in L^∞ , (3.4) and (3.5) with $q = 1$ imply a uniform bound for $|f^e \mathbf{1}_{K^c}(t)|_{\mathcal{E}}$.

We now square (3.13), multiply by Q^{p-2} and integrate over K to get:

$$\begin{aligned} |f(t)|_{L^2(K, d\mu)}^2 &\leq 2|f_0|_{L^2(K, d\mu)}^2 \\ &\quad + C(t) \int_0^t \int_K \left(Q^{p-2} |E_f \cdot \nabla_v Q|^2 \right)(s, X(s; t, x, v), V(s; t, x, v)) ds dx dv, \end{aligned}$$

where we used the fact that $Q(x, v) = Q(X(s; t, x, v), V(s; t, x, v))$. Then, performing again the Lagrangian change of variable (3.10), and remarking from (2.40) that:

$$Q^{p-2} |\nabla_v Q|^2 = \frac{|v| |\nabla_v Q|}{p-1} \in L^\infty(\mathbb{R}_x^N, L^1(\mathbb{R}_v^N)),$$

we obtain:

$$\begin{aligned} |f(t)|_{L^2(K, d\mu)}^2 &\leq 2|f_0|_{L^2(K, d\mu)}^2 + C \int_0^t |E_f|_{L^2(\mathbb{R}^N)}^2(s) ds \\ &\leq 2|f_0|_{L^2(K, d\mu)}^2 + C \int_0^t |f(t)|_{\mathcal{L}\mathcal{E}}^2(s) ds, \end{aligned}$$

where we used (3.1). Since we already have $|f \mathbf{1}_{K^c}|_{\mathcal{E}} \leq C$ for all t , this is enough to conclude with the Gronwall lemma that the function f belongs to $L_{loc}^\infty(\mathbb{R}_+, \mathcal{L}\mathcal{E})$.

Step 4. Derivation of the conservation laws.

It remains to prove the conservation laws (3.3) and (3.6). To this aim, let us first define a suitable regularization of f . Let $n \in \mathbb{N}^*$ and f_0^n be a sequence of $C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N) \cap \mathcal{L}\mathcal{E}$ functions which converges to f_0 in the $\mathcal{L}\mathcal{E}$ topology as $n \rightarrow +\infty$. Consider now a nonnegative $C^\infty(\mathbb{R})$ function θ such that $\theta(u) = 1$ for $u \geq 1$ and $\theta(u) = 0$ for $u \leq 1/2$ and let

$$\theta^n(x, v) = \theta \left(n \left| 1 + \frac{|v|^2}{2} + \phi_Q(x) \right| \right), \quad \chi^n(x, v) = \theta \left(n \left(-1 - \frac{|v|^2}{2} - \phi_Q(x) \right) \right). \quad (3.15)$$

Now, we define f^n as the solution of the following problem, which can be constructed by a fixed-point procedure similarly as above:

$$\partial_t f^n + v \cdot \nabla_x f^n - E_Q \cdot \nabla_v f^n = \theta^n E_{f^n} \cdot \nabla_v Q, \quad f^n(t=0) = f_0^n, \quad (3.16)$$

The function $\theta^n \nabla_v Q$ being C^∞ , it is readily seen that f^n is a sequence of C^∞ function that converges to f^n in the $\mathcal{L}\mathcal{E}$ topology as $n \rightarrow +\infty$.

Now, from (3.16) and (2.40), we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2N}} Q^{p-2} (f^n)^2 dx dv &= 2 \int_K Q^{p-2} f^n \theta^n E_{f^n} \cdot \nabla_v Q dx dv \\ &= -\frac{2}{p-1} \int_K f^n \chi^n v \cdot E_{f^n} dx dv \\ &= -\frac{2}{p-1} \int_K f^n \chi^n v \cdot E_{f^n \chi^n} dx dv - \frac{2}{p-1} \int_K f^n \chi^n v \cdot E_{f^n(1-\chi^n)} dx dv \\ &= \frac{2}{p-1} \int_{\mathbb{R}^N} \phi_{f^n \chi^n} \nabla_x \cdot \left(\int_{\mathbb{R}^N} v f^n \chi^n dv \right) dx - \frac{2}{p-1} \int_K f^n \chi^n v \cdot E_{f^n(1-\chi^n)} dx dv, \end{aligned} \quad (3.17)$$

where we used the equation $v \cdot \nabla_x Q - E_Q \cdot \nabla_v Q = 0$ and remarked that θ^n and χ^n coincide on K . Multiply now (3.16) by χ^n . Since χ^n is a function of the energy $\frac{|v|^2}{2} + \phi_Q(x)$, we have

$$\partial_t (f^n \chi^n) + v \cdot \nabla_x (f^n \chi^n) - E_Q \cdot \nabla_v (f^n \chi^n) = (\chi^n)^2 E_{f^n} \cdot \nabla_v Q. \quad (3.18)$$

Besides, for the same reason, the function $(\chi^n)^2 \nabla_v Q$ is an exact derivative with respect to v . Hence an integration of (3.18) with respect to v yields

$$\partial_t \int_{\mathbb{R}^N} (f^n \chi^n) dv + \nabla_x \cdot \left(\int_{\mathbb{R}^N} v f^n \chi^n dv \right) = 0$$

and, by the Poisson equation, the first integral in the right-hand side of (3.17) can be rewritten as follows:

$$\frac{2}{p-1} \int_{\mathbb{R}^N} \phi_{f^n \chi^n} \nabla_x \cdot \left(\int_{\mathbb{R}^N} v f^n \chi^n dv \right) dx = \frac{1}{p-1} \frac{d}{dt} \int_K \phi_{f^n \chi^n} f^n \chi^n dx dv.$$

It comes finally

$$\begin{aligned} (\mathcal{M}(f^n \chi^n), f^n \chi^n)(t) &= (\mathcal{M}(f_0^n \chi^n), f_0^n \chi^n) \\ &\quad - 2 \int_0^t \int_K f^n(s) \chi^n v \cdot E_{f^n(1-\chi^n)}(s) dx dv ds. \end{aligned}$$

Since, for all $T > 0$, $f^n \rightarrow f$ in $L^\infty((0, T), \mathcal{L}\mathcal{E})$ as $n \rightarrow +\infty$, one can pass to the limit in the various terms of this identity, thanks to Lemma 3.1 (recall that v is bounded on K). Applying this inequality to f^i and f^e leads respectively to (3.3) and (3.6).

This concludes the proof of Proposition 3.2.

Let us conclude this section with the following commutation formula that will be useful in the next subsections:

Lemma 3.4 (Commutation formula) *Let $f_0 \in \mathcal{L}\mathcal{E}$ and $f(t) = e^{-t\mathcal{L}}f_0$ be the corresponding weak solution of (1.23). Let $h \in L^2(K, d\mu) \cap C^1(\overset{\circ}{K})$ be such that $\mathcal{L}h \in L^2(K, d\mu)$, then:*

$$(f(t), \mathcal{M}h) = (f_0, \mathcal{M}h) + \int_0^t (f(s), \mathcal{M}\mathcal{L}h) ds - \int_0^t \int_K v \cdot (hE_{f\mathbf{1}_{K^c}})(s) ds. \quad (3.19)$$

Moreover, for $1 \leq i \leq N$, we have

$$(f(t), x_i \mathbf{1}_K) = (f_0, x_i \mathbf{1}_K) + \int_0^t (f(s), v_i \mathbf{1}_K) ds. \quad (3.20)$$

Proof of Lemma 3.4.

Let us observe the following algebraic identity which follows from a direct computation: let \mathcal{L} , \mathcal{T} , \mathcal{M} be respectively given by (1.24), (3.8) and (1.21), then:

$$\forall f \in L^2(K, d\mu), \quad \mathcal{L}f = \frac{1}{(p-1)Q^{p-2}} \mathcal{T}(\mathcal{M}f) \quad (3.21)$$

in the sense of distributions.

Introduce then the smooth function θ^n defined by (3.15) and let \widetilde{f}_0^n be a C_0^∞ regularization of f_0 . Then, setting $f_0^n = \widetilde{f}_0^n \theta^n$, we define f^n as the classical solution of (3.16). One can see that $f^n \rightarrow f$ in $\mathcal{L}\mathcal{E}$ as $n \rightarrow +\infty$. Moreover, since the flow (3.9) preserves the energy and θ^n only depends on the energy, the support of f^n is included in the support of θ^n , where h is C^1 . We split $f^n = f_i^n \mathbf{1}_K + f_e^n \mathbf{1}_{K^c} = f_i^n + f_e^n$ with $f_{i,e}^n$ smooth from the support localization of f^n , and thus:

$$\frac{d}{dt} (f^n, \mathcal{M}h) = -(\mathcal{L}f^n, \mathcal{M}h) = -(\mathcal{L}f_i^n, \mathcal{M}h) - (\mathcal{L}f_e^n, \mathcal{M}h).$$

The first term is computed using (3.21), the self-adjointness of \mathcal{M} , the skew-adjointness of \mathcal{T} and the fact that $\mathcal{T}(F(e)) = 0$ for any function F :

$$\begin{aligned} -(\mathcal{L}f_i^n, \mathcal{M}h) &= -\left(\frac{1}{(p-1)Q^{p-2}} \mathcal{T}(\mathcal{M}f_i^n), \mathcal{M}h\right) = (\mathcal{M}f_i^n, \mathcal{T}\left[\frac{1}{(p-1)Q^{p-2}} \mathcal{M}h\right]) \\ &= (\mathcal{M}f_i^n, \mathcal{L}h) = (f_i^n, \mathcal{M}\mathcal{L}h) = (f^n, \mathcal{M}\mathcal{L}h). \end{aligned}$$

For the second term, we use that f_e^n and $\mathcal{M}h$ have disjoint support to compute:

$$-(\mathcal{L}f_e^n, \mathcal{M}h) = \int_{\mathbb{R}^N} E_{f_e^n} \cdot \nabla_v Q (p-1) Q^{p-2} h = - \int_K h v \cdot E_{f_e^n}$$

from (2.40). We then integrate in time and pass to the limit as $n \rightarrow +\infty$, and (3.19) follows. The second identity (3.20) can be proved by a similar regularization procedure and by direct calculations. This concludes the proof of Lemma 3.4.

Remark 3.5 *If the support of f_0 is in K then (3.19) becomes simpler:*

$$(f(t), \mathcal{M}h) = (f_0, \mathcal{M}h) + \int_0^t (f(s), \mathcal{M}\mathcal{L}h) ds. \quad (3.22)$$

3.2 The linearized dynamics on the support of Q

From Proposition 3.2, the solution $e^{-t\mathcal{L}}f_0$ of (1.23) remains supported on K when it has this property at $t = 0$. In this section we estimate the action of the linearized Vlasov-Poisson system on $L^2(K, d\mu)$.

Let us start by introducing the following decomposition of $L^2(K, d\mu)$ whose proof is given in the Appendix.

Lemma 3.6 (Decomposition of $L^2(K, d\mu)$) *Let $(N, \alpha) \in \{(3, 1), (3, 2), (4, 2)\}$. There holds the decomposition*

$$L^2(K, d\mu) = M \oplus S,$$

where M is defined as the set of $f \in L^2(K, d\mu)$ with

$$\left\{ \begin{array}{l} \text{for } N \neq \alpha + 2, \quad \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) = (f, x_i) = (f, v_i) = 0, \quad 1 \leq i \leq N, \\ \text{for } N = \alpha + 2, \quad \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) = (f, x \cdot v |v|^{\alpha-2}) = (f, |x|^2 |v|^{2-\alpha}) = (f, x_i) = (f, v_i) = 0, \\ \hspace{25em} 1 \leq i \leq N, \end{array} \right.$$

and S is defined thanks to the functions S_i given by (2.31) according to:

$$\left\{ \begin{array}{l} \text{if } N \neq \alpha + 2 \text{ then } S = \text{span} \{S_1, \partial_{x_i} Q, \partial_{v_i} Q, 1 \leq i \leq N\}, \\ \text{if } N = \alpha + 2 \text{ then } S = \text{span} \{S_1, S_2, S_3, \partial_{x_i} Q, \partial_{v_i} Q, 1 \leq i \leq N\}. \end{array} \right.$$

Our main claim is now that S and M are invariant under the linearized flow (1.23). The subspace S is the so-called ‘‘flag space’’ and contains the algebraically growing modes induced by the large set of symmetries (1.7), while the free evolution remains bounded on M in the $\mathcal{L}\mathcal{E}$ norm.

Proposition 3.7 (Splitting of the motion) *Let $(N, \alpha) \in \{(3, 2), (4, 2)\}$. Consider the decomposition $L^2(K, d\mu) = M \oplus S$, where the spaces M, S are defined in Lemma 3.6. Then M and S are both invariant under the linearized flow (1.23) and there holds: $\forall t \in \mathbb{R}_+$,*

$$\forall g_0 \in M, \quad \left| e^{-t\mathcal{L}} g_0 \right|_{L^2(K, d\mu)} \leq C |g_0|_{L^2(K, d\mu)}, \quad (3.23)$$

$$\forall g_0 \in S, \quad \left| e^{-t\mathcal{L}} g_0 \right|_{L^2(K, d\mu)} \leq \begin{cases} C(1+t) |g_0|_{L^2(K, d\mu)} & \text{for } N = 3, \\ C(1+t^2) |g_0|_{L^2(K, d\mu)} & \text{for } N = 4. \end{cases} \quad (3.24)$$

Proof of Proposition 3.7

Step 1. The evolution on S .

The free evolution is explicit on S . Indeed, let $N = 3, 4$ and $(S_i)_{1 \leq i \leq 3}$ defined by (2.31), we claim:

$$\mathcal{L}S_1 = 0 \quad \mathcal{L}(\partial_{x_i}Q) = 0, \quad \mathcal{L}(\partial_{v_i}Q) = -\partial_{x_i}Q \quad (3.25)$$

and the extra relations for $N = 4$:

$$\mathcal{L}S_2 = S_1, \quad \mathcal{L}S_3 = S_2. \quad (3.26)$$

Proof of (3.25) and (3.26): They follow from (2.32) and (3.21). Let us prove (3.26). In the interior of K , $(S_i)_{1 \leq i \leq 3}$ are smooth. We then compute using (2.32): $\forall (x, v) \in \overset{\circ}{K}$,

$$\mathcal{L}S_2 = \frac{1}{(p-1)Q^{p-2}} \mathcal{J}(\mathcal{M}S_2) = \frac{1}{(p-1)Q^{p-2}} \mathcal{J}(x \cdot v) = \frac{1}{(p-1)Q^{p-2}} (|v|^2 - x \cdot E_Q).$$

We now take the derivative of (2.33) in x and v to get:

$$E_Q + (p-1)Q^{p-2} \nabla_x Q = 0, \quad v + (p-1)Q^{p-2} \nabla_v Q = 0$$

and thus

$$\mathcal{L}S_2 = \frac{1}{(p-1)Q^{p-2}} (|v|^2 - x \cdot E_Q) \mathbf{1}_K = -v \cdot \nabla_v Q + x \cdot \nabla_x Q = S_1.$$

Similarly,

$$\begin{aligned} \mathcal{L}S_3 &= \frac{1}{(p-1)Q^{p-2}} \mathcal{J}(\mathcal{M}S_3) = \frac{1}{(p-1)Q^{p-2}} \mathcal{J}\left(\frac{|x|^2}{2}\right) \\ &= \frac{x \cdot v}{(p-1)Q^{p-2}} = -x \cdot \nabla_v Q = S_2 \end{aligned}$$

where we used (2.40). In order to conclude the proof of (3.26), we use the following technical remark. Let $h \in L^2(K, d\mu)$ such that $h \in C^1(\overset{\circ}{K})$ and denote by $(\mathcal{J}h)_K$ the function defined pointwise on $\overset{\circ}{K}$ and continued by zero outside K . Assume that $(\mathcal{J}h)_K \in L^1(K)$. Then, due to the fact that the boundary of K is a level set of the microscopic energy, the distribution $\mathcal{J}h$ defined as the distributional derivative of h by the derivation \mathcal{J} and $(\mathcal{J}h)_K$ coincide in $\mathcal{D}'(\mathbb{R}^N \times \mathbb{R}^N)$. Applying this with $h = S_1$ or S_2 concludes the proof of (3.26). Next (3.25) follows similarly and is left to the reader.

Let now $g_0 \in S$, i.e. according to Lemma 3.6:

$$\begin{aligned} \text{if } N = 3, \quad g_0 &= \alpha S_1 + \sum_{i=1}^N \delta_i \partial_{x_i} Q + \sum_{i=1}^N \epsilon_i \partial_{v_i} Q, \\ \text{if } N = 4, \quad g_0 &= \alpha S_1 + \beta S_2 + \gamma S_3 + \sum_{i=1}^N \delta_i \partial_{x_i} Q + \sum_{i=1}^N \epsilon_i \partial_{v_i} Q. \end{aligned}$$

From (3.25) and (3.26), the evolution $e^{-t\mathcal{L}}g_0$ is explicit:

$$\begin{aligned} \text{if } N = 3, \quad e^{-t\mathcal{L}}g_0 &= \alpha S_1 + \sum_{i=1}^N (\epsilon_i t + \delta_i) \partial_{x_i} Q + \sum_{i=1}^N \epsilon_i \partial_{v_i} Q, \\ \text{if } N = 4, \quad e^{-t\mathcal{L}}g_0 &= \left(\frac{\gamma}{2}t^2 - \beta t + \alpha\right) S_1 + (-\gamma t + \beta) S_2 + \gamma S_3 + \sum_{i=1}^N (\epsilon_i t + \delta_i) \partial_{x_i} Q + \sum_{i=1}^N \epsilon_i \partial_{v_i} Q, \end{aligned}$$

which shows the stability of S and (3.24) is proved.

Step 2. The evolution on M .

Let now $g_0 \in M$. From (3.25) and (3.26), we have $\mathcal{M}\mathcal{L}S_1 = 0$ and for $N = 4$, $\mathcal{M}\mathcal{L}S_2 = \mathcal{M}S_1$, $\mathcal{M}\mathcal{L}S_3 = \mathcal{M}S_2$. Hence Lemma 3.4 and Remark 3.5 give

$$\frac{d}{dt}(e^{-t\mathcal{L}}g_0, \mathcal{M}S_1) = 0, \quad (3.27)$$

and, if $N = 4$,

$$\frac{d}{dt}(e^{-t\mathcal{L}}g_0, \mathcal{M}S_2) = (e^{-t\mathcal{L}}g_0, \mathcal{M}S_1), \quad \frac{d}{dt}(e^{-t\mathcal{L}}g_0, \mathcal{M}S_3) = (e^{-t\mathcal{L}}g_0, \mathcal{M}S_2). \quad (3.28)$$

Furthermore, remarking that $\mathcal{M}\partial_{v_i}Q = -v_i$, we deduce from (3.25), (3.22), (3.20) and (2.30) that for $1 \leq i \leq N$:

$$\frac{d}{dt}(e^{-t\mathcal{L}}g_0, v_i) = (e^{-t\mathcal{L}}g_0, \mathcal{M}\partial_{x_i}Q) = 0, \quad \frac{d}{dt}(e^{-t\mathcal{L}}g_0, x_i) = (e^{-t\mathcal{L}}g_0, v_i). \quad (3.29)$$

Recalling from Lemma 3.6 and from (2.32) that M can be characterized as the set of $f \in L^2(K, d\mu)$ with:

$$\begin{cases} \text{for } N = 3, & (f, \mathcal{M}S_1) = (f, x_i) = (f, v_i) = 0, \quad 1 \leq i \leq N, \\ \text{for } N = 4, & (f, \mathcal{M}S_j) = (f, x_i) = (f, v_i) = 0, \quad 1 \leq j \leq 3, \quad 1 \leq i \leq N, \end{cases}$$

we infer from (3.27), (3.28) and (3.29) that $g_0 \in M$ implies $e^{-t\mathcal{L}}g_0 \in M$ for all $t \geq 0$. The uniform bound (3.23) on $e^{-t\mathcal{L}}g$ in $\mathcal{L}\mathcal{E}$ now follows from the conservation of the linearized energy-Casimir functional (3.3) and the coercivity property of Theorem 1.3 which ensures that the quadratic form $(\mathcal{M}\cdot, \cdot)$ is coercive on M . This concludes the proof of Proposition 3.7.

3.3 Proof of Theorem 1.4

We are now in position to conclude the proof of Theorem 1.4.

Proof of Theorem 1.4

Let $f_0 = f_0^i + f_0^e \in \mathcal{L}\mathcal{E}$ according to the decomposition (3.2) and denote respectively $f^i(t) = e^{-t\mathcal{L}}f_0^i$, $f^e(t) = e^{-t\mathcal{L}}f_0^e$ the corresponding solutions to (1.23). The evolution of f^i is already controlled thanks to Proposition 3.7. It remains to study the evolution of f^e which is not supported a priori in K^c and may spread onto the whole space.

From Proposition 3.2, we have already

$$|f^e \mathbf{1}_{K^c}|_{\mathcal{E}}(t) \leq C |f_0^e|_{\mathcal{E}} = C |f_0^e|_{\mathcal{L}\mathcal{E}}.$$

In particular,

$$|E_{f^e \mathbf{1}_{K^c}}|_{L^2} \leq C |f_0^e|_{\mathcal{L}\mathcal{E}}. \quad (3.30)$$

It remains to bound $f^e \mathbf{1}_K$. Using Lemma 3.4 with $h = S_1$, $h = S_2$, $h = S_3$ and $h = \partial_{v_i} Q$ (the S_i 's are defined in (2.31)), we obtain successively:

$$\left| \left(f^e, \left(\frac{|v|^2}{2} + \phi_Q \right) \mathbf{1}_K \right) \right| (t) \leq C(1+t) |f_0^e|_{\mathcal{L}\mathcal{E}},$$

$$|(f^e, v_i \mathbf{1}_K)| (t) \leq C(1+t) |f_0^e|_{\mathcal{L}\mathcal{E}},$$

$$|(f^e, x_i \mathbf{1}_K)| (t) \leq C(1+t^2) |f_0^e|_{\mathcal{L}\mathcal{E}},$$

and, if $N = 4$,

$$|(f^e, x \cdot v \mathbf{1}_K)| (t) \leq C(1+t^2) |f_0^e|_{\mathcal{L}\mathcal{E}},$$

$$|(f^e, |x|^2 \mathbf{1}_K)| (t) \leq C(1+t^3) |f_0^e|_{\mathcal{L}\mathcal{E}},$$

where we applied Lemma 2.4 and also used (3.30) to bound the various terms $\int_K h v \cdot E_{\mathbf{1}_{K^c}}(s) ds$. Therefore, Theorem 1.3 implies that

$$|f^e(t)|_{L^2(K, d\mu)}^2 \leq C |(\mathcal{M}f^e, f^e)| (t) + C(1+t^{2\alpha}) |f_0^e|_{\mathcal{L}\mathcal{E}}^2,$$

with $\alpha = 2$ if $N = 3$ and $\alpha = 3$ if $N = 4$. Now, one deduces from (3.6) and (3.30) that

$$\begin{aligned} |f^e(t)|_{L^2(K, d\mu)}^2 &\leq C |(\mathcal{M}f^e, f^e)| (t) + C(1+t^{2\alpha}) |f_0^e|_{\mathcal{L}\mathcal{E}}^2 \\ &\leq C(1+t^{2\alpha}) |f_0^e|_{\mathcal{L}\mathcal{E}}^2 + C |f_0^e|_{\mathcal{L}\mathcal{E}} \int_0^t |f^e(s)|_{L^2(K, d\mu)} ds, \end{aligned}$$

and (1.26) follows from a standard sublinear Gronwall Lemma. This concludes the proof of Theorem 1.4.

Appendix

This Appendix is devoted to the proof of the orthogonal decomposition $L^2(K, d\mu) = M \oplus S$ of Lemma 3.6.

Proof of Lemma 3.6.

Explicit computations using the identities

$$\| |v|^\alpha Q \|_{L^1} = \frac{N-2}{2} \|E_Q\|_{L^2}^2 = \int_{\mathbb{R}^{2N}} x \cdot \nabla_x \phi_Q Q \, dx dv$$

lead to

$$\left(\frac{|v|^\alpha}{\alpha} + \phi_Q, S_1 \right) = (N - \alpha - 2) \frac{(2\alpha + (\alpha + 2)N - N^2)}{4\alpha} \|E_Q\|_{L^2}^2. \quad (\text{A.1})$$

Moreover, integrations by parts yield

$$(x_i, \partial_{x_j} Q) = (v_i, \partial_{v_j} Q) = -\delta_{ij} \int Q, \quad 1 \leq i \leq N, \quad 1 \leq j \leq N, \quad (\text{A.2})$$

and the radial symmetry of Q implies that

$$0 = \left(\frac{|v|^\alpha}{\alpha} + \phi_Q, S_2 \right) = \left(\frac{|v|^\alpha}{\alpha} + \phi_Q, \partial_{x_i} Q \right) = \left(\frac{|v|^\alpha}{\alpha} + \phi_Q, \partial_{v_i} Q \right), \quad 1 \leq i \leq N, \quad (\text{A.3})$$

$$(x_i, S_j) = 0, \quad 1 \leq i \leq N, \quad 1 \leq j \leq 3. \quad (\text{A.4})$$

Let $N \neq \alpha + 2$ and $f \in L^2(K, d\mu)$. We look for $\lambda, (\delta_i)_{1 \leq i \leq N}, (\epsilon_i)_{1 \leq i \leq N}$ such that

$$\tilde{f} = f - \lambda S_1 - \sum_{i=1}^N \delta_i \partial_{x_i} Q - \sum_{i=1}^N \epsilon_i \partial_{v_i} Q \in M.$$

Taking the inner product of \tilde{f} with $\frac{|v|^\alpha}{\alpha} + \phi_Q$ and using (A.1) and (A.3), we get

$$\lambda = \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) \left[(N - \alpha - 2) \frac{(2\alpha + (\alpha + 2)N - N^2)}{4\alpha} |E_Q|_{L^2}^2 \right]^{-1}.$$

Taking the inner product of \tilde{f} with x_i or v_i and using (A.4) and (A.2), we get

$$\delta_i = (f, x_i) \left[- \int Q \right]^{-1}, \quad \epsilon_i = (f, v_i) \left[- \int Q \right]^{-1}, \quad 1 \leq i \leq N.$$

Since λ , the δ_i 's and the ϵ_i 's are uniquely defined in order to ensure $\tilde{f} \in M$, the result is proved.

Let $N = \alpha + 2$. By explicit computations and using the symmetries of Q , we get:

$$\alpha \left(\frac{|v|^\alpha}{\alpha} + \phi_Q, S_3 \right) = (\mathcal{M}S_1, S_3) = (S_1, \mathcal{M}S_3) = \left(S_1, \frac{|v|^{2-\alpha}|x|^2}{2} \right) = -\frac{\alpha}{2} \int |v|^{2-\alpha}|x|^2 Q, \quad (\text{A.5})$$

$$(|v|^{2-\alpha}|x|^2, S_2) = (|v|^{2-\alpha}|x|^2, \partial_{x_i} Q) = (|v|^{2-\alpha}|x|^2, \partial_{v_i} Q) = 0, \quad 1 \leq i \leq N, \quad (\text{A.6})$$

$$(x \cdot v |v|^{\alpha-2}, S_1) = (x \cdot v |v|^{\alpha-2}, \partial_{x_i} Q) = (x \cdot v |v|^{\alpha-2}, \partial_{v_i} Q) = 0, \quad 1 \leq i \leq N, \quad (\text{A.7})$$

$$(x \cdot v |v|^{\alpha-2}, S_3) = (\mathcal{M}S_2, S_3) = (S_2, \mathcal{M}S_3) = 0 \quad (\text{A.8})$$

$$(x \cdot v |v|^{\alpha-2}, S_2) = \int |x|^2 Q + (\alpha - 2) \int (x \cdot v)^2 |v|^{\alpha-4} Q. \quad (\text{A.9})$$

Let $f \in L^2(K, d\mu)$. We deduce from (A.1)–(A.9) that we have

$$\tilde{f} = f - \lambda S_1 - \beta S_2 - \gamma S_3 - \sum_{i=1}^N \delta_i \partial_{x_i} Q - \sum_{i=1}^N \epsilon_i \partial_{v_i} Q \in M$$

if, and only if:

$$\gamma = \left(f, \frac{|v|^\alpha}{\alpha} + \phi_Q \right) \left[-\frac{1}{2} \int |v|^{2-\alpha}|x|^2 Q \right]^{-1},$$

$$\lambda = [(f, |v|^{2-\alpha}|x|^2) - \gamma (S_3, |v|^{2-\alpha}|x|^2)] \left[-\alpha \int |v|^{2-\alpha}|x|^2 Q \right]^{-1},$$

$$\beta = (f, x \cdot v |v|^{\alpha-2}) \left[\int |x|^2 Q + (\alpha - 2) \int (x \cdot v)^2 |v|^{\alpha-4} Q \right]^{-1},$$

$$\delta_i = (f, x_i) \left[- \int Q \right]^{-1}, \quad \epsilon_i = (f, v_i) \left[- \int Q \right]^{-1}, \quad 1 \leq i \leq N.$$

where we have successively taken the inner product of \tilde{f} with $\frac{|v|^\alpha}{\alpha} + \phi_Q$, $|v|^{2-\alpha}|x|^2$, $x \cdot v|v|^{\alpha-2}$, x_i and v_i . This concludes the proof of Lemma 3.6. \square

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